Financial Economics: Risk Sharing and Asset Pricing in General Equilibrium II ©

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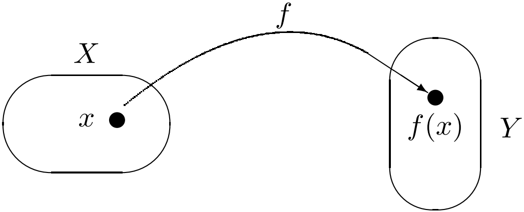
1. Math I: Basics

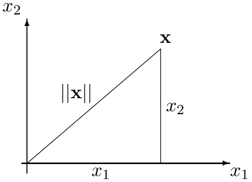
Definition: Let X and Y be two sets. A rule that maps each element x of X to an element $f(x)$ of Y is a ***function***. A convenient shorthand is $f : X \to Y$.



Definition: Let X be a set. A ***sequence*** is a mapping from the natural numbers $\mathbb{N} = 1, 2, 3, \ldots$ to the elements of X. The n-th element of the sequence is denoted x_n.

Definition: Let $\mathbf{x} = (x_1, \ldots, x_L)$ be an L-tuple of real numbers. The set of all such L-tuples is denoted as $\mathbb{R}^L$ and the subsets with $x_l \geq 0$ or $x_l > 0$ for all $l = 1, \ldots, L$ as $\mathbb{R}^L_+$ and $\mathbb{R}^L_{++}$, respectively. The ***(Euclidian) distance*** between $\mathbf{x} \in \mathbb{R}^L$ and $\mathbf{y} \in \mathbb{R}^L$ is defined as

$$||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{l=1}^{L} (x_l - y_l)^2}.$$



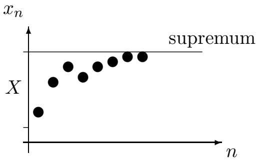
Definition: A sequence of elements of a subset X of $\mathbb{R}^L$ is said to ***converge*** to $\mathbf{a}$ if for any real number $\epsilon > 0$, there is n_ϵ such that

$$||\mathbf{x}_n - \mathbf{a}|| < \epsilon \quad \text{for } n \geq n_\epsilon.$$

A convenient shorthand is $\mathbf{x}_n \to \mathbf{a}$.

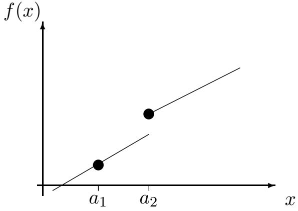
Definition: The least upper bound of a set $X \subset \mathbb{R}$ is called its ***supremum***.

We take the following for granted: Let $X \subset \mathbb{R}$ be bounded from above. Then a supremum exists, and a sequence $x_n \in X$ converging to the supremum exists.



This fact can be derived from an axiom of continuity for the real number system.

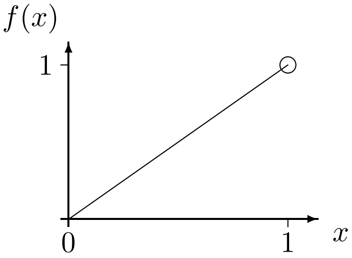
Definition: Let $X, Y \subset \mathbb{R}^L$ and $\mathbf{f} : X \to Y$. $\mathbf{f}$ is continuous at $\mathbf{a} \in X$ if for any sequence $\mathbf{x}_n \to \mathbf{a}$, we have $\mathbf{f}(\mathbf{x}_n) \to \mathbf{f}(\mathbf{a})$. $\mathbf{f}$ is ***continuous*** if it is continuous for all $\mathbf{a} \in X$.



Definition: $X \subset \mathbb{R}^L$ is ***closed*** if for any convergent sequence $\mathbf{x}_n \in X$ the limit $\mathbf{a}$ is in X.
Definition: $X \subset \mathbb{R}^L$ is ***bounded*** if there exist $\mathbf{a}, \mathbf{b} \in \mathbb{R}^L$ such that

$$X \subset Y = \{\mathbf{x} | \, \mathbf{x} \in \mathbb{R}^L, a_l \leq x_l \leq b_l \text{ for } l = 1, \ldots, L\}.$$

Definition: A set $X \subset \mathbb{R}^L$ is ***compact*** if a convergent subsequence $\mathbf{x}_m$ can be extracted from any sequence $\mathbf{x}_n \in X$.
Compactness of the domain is the crucial condition in order for a continuous function to attain a maximum and is implied by closedness and boundedness.



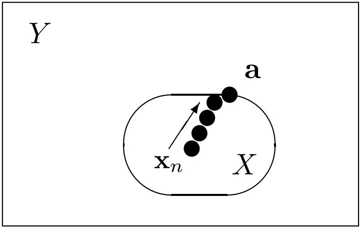
- $f(x) = x : [0, 1) \to \mathbb{R}$ does not attain a maximum, because the domain is not closed.
- $f(x) = x : \mathbb{R}_+ \to \mathbb{R}$ does not attain a maximum, because the domain is not bounded.
- $f(x) = x : [0, 1] \to \mathbb{R}$ attains a maximum $f(1) = 1$ at $x = 1$.

The interval $[a, b] \subset \mathbb{R}$ is compact. Consider a sequence $x_n \in [a, b]$. An infinite number of terms is either in $[a, (a+b)/2]$ or in $[(a+b)/2, b]$. Pick this subinterval. Again, an infinite number of terms of x_n is in the lower or upper half. After m iterations, an infinite number of terms is in a subinterval of length $(b-a)2^{-m}$. Taking the limit $m \to \infty$ yields the limit of a convergent subsequence.

$Y = \{\mathbf{x} | \, \mathbf{x} \in \mathbb{R}^L, a_l \leq x_l \leq b_l \text{ for } l = 1, \ldots, L\}$ is compact. Consider the sequence x_{1n} of the first component of a sequence $\mathbf{x}_n \in Y$. A convergent subsequence x_{1m} exists. Consider the sequence $\mathbf{x}_m$ that includes the other components. The sequence x_{2m} of second components has a convergent subsequence. Repeating the procedure proves the assertion.

Theorem: *If $X \subset \mathbb{R}^L$ is closed and bounded, then it is compact.*

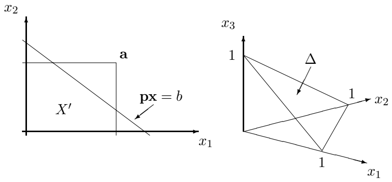
Proof: Since X is bounded, there exist $\mathbf{a}$, $\mathbf{b}$ such that $X \subset Y$. Since Y is compact, it suffices to prove the following assertion: If $X \subset Y$ is closed and Y is compact, X is compact.



Consider a sequence $\mathbf{x}_n$ in X. As $X \subset Y$, this is also a sequence in Y. Since Y is compact, there is a convergent subsequence $\mathbf{x}_m$ with a limit $\mathbf{a}$ in Y. Since $\mathbf{x}_m \to \mathbf{a}$, $\mathbf{x}_m \in X$ and X is closed, we have $\mathbf{a} \in X$. That is, there is the subsequence $\mathbf{x}_m$ extracted from $\mathbf{x}_n \in X$ converges to $\mathbf{a} \in X$. This proves that X is compact. Q.E.D.

Let $\mathbf{a} \in \mathbb{R}^L_+$, $\mathbf{p} \in \mathbb{R}^L_+$, $w > 0$. Consider the following sets:

- $X = \{\mathbf{x} | \, \mathbf{x} \in \mathbb{R}^L_+, \mathbf{px} \leq w\}$ for $\mathbf{p} \in \mathbb{R}^L_{++}$,
- $X' = X \cap \{\mathbf{x} | \, \mathbf{x} \leq \mathbf{a}\}$ for all $\mathbf{p} \in \mathbb{R}^L_+$,
- the unit simplex $\Delta \equiv \{\mathbf{x} | \sum_{l=1}^L x_l = 1, x_l \geq 0, l = 1, \ldots, L\}$.



These sets are obviously bounded. We take it for granted that they are closed, as they contain their boundaries. So they are compact.

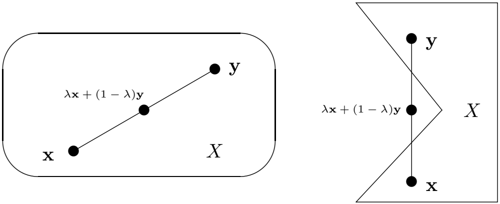
Theorem (Weierstrass Theorem): *Let $X \subset \mathbb{R}^L$ be compact and $\mathbf{f} : X \to \mathbb{R}$ continuous. Then $\mathbf{f}$ takes on a maximum in X.*

Proof: Denote the supremum of the set $\{f(\mathbf{x}) | \, \mathbf{x} \in X\}$ as α. There is a sequence $\mathbf{x}_n$ of elements of X such that $f(\mathbf{x}_n) \to \alpha$. Because of compactness of X, a convergent subsequence $\mathbf{x}_m$ can be extracted from $\mathbf{x}_n$, which converges to $\mathbf{a} \in X$. $f(\mathbf{x}_m)$ is a subsequence of $f(\mathbf{x}_n)$, so $f(\mathbf{x}_m) \to \alpha$. Because of continuity of f, $f(\mathbf{x}_m) \to f(\mathbf{a})$. So $\alpha = f(\mathbf{a})$. That is, (the finite value) $f(\mathbf{a})$ is the maximum of f on X. Q.E.D.

2. Math II: Separation of convex sets

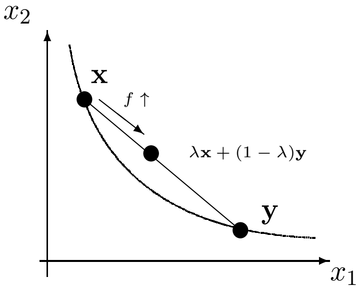
Definition: A set $X \subset \mathbb{R}^L$ is ***convex*** if for all $\mathbf{x}, \mathbf{y} \in X$,

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in X \quad \text{for } \lambda \in [0, 1].$$



Definition: Let $X \subset \mathbb{R}_L$ be convex. $f : X \to \mathbb{R}$ is ***strictly quasi-concave*** if for all $\mathbf{x}, \mathbf{y} \in X$ ($\mathbf{y} \neq \mathbf{x}$),

$$f(\mathbf{x}) \geq f(\mathbf{y}) \Rightarrow f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) > f(\mathbf{y}) \quad \text{for } \lambda \in (0,1).$$



Theorem (Alternative characterization of quasi-concavity): *Let $X \subset \mathbb{R}_L$ be convex and $f : X \to \mathbb{R}$ strictly quasi-concave. Then for given a, the set*

$$F = \{\mathbf{x} | \, \mathbf{x} \in X, f(\mathbf{x}) \geq a\}$$

is convex.

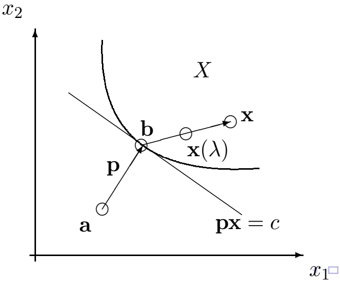
Proof: Consider $\mathbf{x}', \mathbf{x}'' \in F$ with $\mathbf{x}' \neq \mathbf{x}''$ and (without loss of generality) $f(\mathbf{x}') \geq f(\mathbf{x}'')$. $f(\mathbf{x}') \geq f(\mathbf{x}'') \geq a$ because $\mathbf{x}', \mathbf{x}'' \in F$. From quasi-concavity of f,

$$f(\lambda \mathbf{x}' + (1 - \lambda)\mathbf{x}'') > f(\mathbf{x}'') \geq a, \quad \lambda \in (0, 1),$$

so $\lambda \mathbf{x}' + (1 - \lambda)\mathbf{x}'' \in F$. Q.E.D.

Definition: Let $\mathbf{p} \in \mathbb{R}^L$ and $c \in \mathbb{R}$. The set $\{\mathbf{x} | \mathbf{x} \in \mathbb{R}^L, \mathbf{px} = c\}$ is called a ***hyperplane***.

Theorem (Separating hyperplane theorem 1): Let $X \subset \mathbb{R}^L$ be convex and closed, $\mathbf{a} \in \mathbb{R}^L$, and $\mathbf{a} \notin X$. Then there exist $\mathbf{p} \in \mathbb{R}^L$ and $c \in \mathbb{R}$ such that $\mathbf{pa} < c$ and $\mathbf{px} \geq c$ for all $\mathbf{x} \in X$. That is, there exists a hyperplane that separates $\mathbf{a}$ and X.



Proof: Since X is closed, the distance between **a** and X is positive. Let **b** be the point in X closest to **a.** **b** exists due to the Weierstrass theorem. Let $\mathbf{p} = \mathbf{b} - \mathbf{a}$ and $c = \mathbf{pb}$. Notice

$$\mathbf{pa} = \mathbf{p}(\mathbf{a} - \mathbf{b}) + \mathbf{pb} = -||\mathbf{a} - \mathbf{b}||^2 + \mathbf{pb} < \mathbf{pb} = c.$$

Because of convexity, for any $\mathbf{x} \in X$, any point $\mathbf{x}(\lambda) = \lambda\mathbf{x} + (1 - \lambda)\mathbf{b} \in X$. The distance between $\mathbf{x}(\lambda)$ and **a** is

$$||\mathbf{x}(\lambda) - \mathbf{a}|| = \sqrt{[(\mathbf{b} - \mathbf{a}) + \lambda(\mathbf{x} - \mathbf{b})]^2}.$$

By construction $||\mathbf{x}(\lambda) - \mathbf{a}|| \geq ||\mathbf{b} - \mathbf{a}||$:

$$\sqrt{[(\mathbf{b} - \mathbf{a}) + \lambda(\mathbf{x} - \mathbf{b})]^2} \geq \sqrt{(\mathbf{b} - \mathbf{a})^2}.$$

Rearranging terms:

$$2\lambda(\mathbf{b}-\mathbf{a})(\mathbf{x}-\mathbf{b})+\lambda^2(\mathbf{x}-\mathbf{b})^2 \geq 0.$$

Dividing by λ (> 0), letting $\lambda \to 0$, and using $\mathbf{p} = \mathbf{b} - \mathbf{a}$ and $\mathbf{pb} = c$ shows

$$\mathbf{px} \geq c \quad \text{for } \mathbf{x} \in X.$$

That is, the hyperplane $\mathbf{px} = c$ separates **a** and X. The hyperplane passes through **b**. There are hyperplanes through points between **a** and **b** which strictly separate **a** and X, i.e., $\mathbf{px} > c'$ for $\mathbf{x} \in X$. Q.E.D

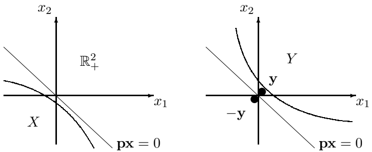
Theorem (Separating hyperplane theorem 2): Let $X \subset \mathbb{R}^L$ be convex, $\mathbf{a} \in \mathbb{R}^L$, and $\mathbf{a} \notin X$. Then there exist $\mathbf{p} \in \mathbb{R}^L$ and $c \in \mathbb{R}$ such that $\mathbf{pa} < c$ and $\mathbf{px} \geq c$ for all $\mathbf{x} \in X$. That is, there exists a hyperplane that separates $\mathbf{a}$ and X.

Proof: If **a** is not a boundary point of X, the proof of the former theorem goes through with minor modifications. If **a** is a boundary point of X, then there are points close to **a** not in X and hyperplanes strictly separating these points and X. Consider a sequence $\mathbf{a}_n$ of such points and the corresponding hyperplanes $\mathbf{p}_n\mathbf{x} = \mathbf{p}_n\mathbf{b}_n = c'_n$. By construction, $\mathbf{p}_n\mathbf{a}_n < c'_n$ and $\mathbf{p}_n\mathbf{x} > c'_n$ for $\mathbf{x} \in X$. Taking the limit, $\mathbf{pa} = c'$ and $\mathbf{px} \geq c'$ for $\mathbf{x} \in X$. That is, there is a hyperplane through **a** that "supports" X. Q.E.D.

Theorem (Separating hyperplane theorem 3): Let $X \subset \mathbb{R}^L$ be convex and disjoint from $\mathbb{R}^L_{++}$. Then X and $\mathbb{R}^L_+$ are separated by the hyperplane $\mathbf{px} = 0$ with $\mathbf{p} \geq \mathbf{0}$ so that $\mathbf{px} \leq 0$ for all $\mathbf{x} \in X$.

Proof: Let $Y = \mathbb{R}^L - X$, i.e.,

$$Y = \{\mathbf{y} | \, \mathbf{y} = \mathbf{z} - \mathbf{x}, \mathbf{z} \in \mathbb{R}^L, \mathbf{x} \in X\}.$$



Consider $\mathbf{y}', \mathbf{y}'' \in Y$. There are $\mathbf{z}', \mathbf{z}'' \in \mathbb{R}^L_+$ and $\mathbf{x}', \mathbf{x}'' \in X$ such that $\mathbf{y}' = \mathbf{z}' - \mathbf{x}'$ and $\mathbf{y}'' = \mathbf{z}'' - \mathbf{x}''$. By convexity of $\mathbb{R}^L_+$ and X, $\lambda\mathbf{z}' + (1-\lambda)\mathbf{z}'' \in \mathbb{R}^L_+$ and $\lambda\mathbf{x}' + (1-\lambda)\mathbf{x}'' \in X$ for $\lambda \in [0,1]$. Hence, $\lambda(\mathbf{z}' - \mathbf{x}') + (1-\lambda)(\mathbf{z}'' - \mathbf{x}'') = \lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'' \in Y$, i.e., Y is convex.

Suppose the origin $\mathbf{0}$ is in the interior of Y. Then there is $\mathbf{y} > \mathbf{0}$, $\mathbf{y} \in Y$, and also $-\mathbf{y} \in Y$. So $-\mathbf{y} = \mathbf{z} - \mathbf{x}$ for some $\mathbf{z} \in \mathbb{R}^L$, $\mathbf{x} \in X$. Hence,

$$\mathbf{0} < \mathbf{y} = \mathbf{x} - \underbrace{\mathbf{z}}_{\geq \mathbf{0}} \leq \mathbf{x}.$$

This contradicts the assumption that X and $\mathbb{R}^L_{++}$ are disjoint. So $\mathbf{0}$ is not in the interior of Y

By separating hyperplane theorem 2, there is a hyperplane $\mathbf{py} = 0$ such that

$$\mathbf{py} \geq 0 \quad \text{for } y \in Y.$$

Using $\mathbf{y} = \mathbf{z} - \mathbf{x}$,

$$\mathbf{pz} \geq \mathbf{px} \quad \text{for } z \in \mathbb{R}^L_+, \mathbf{x} \in X.$$

Set $\mathbf{z} = \mathbf{0} \in \mathbb{R}^L_+$ to get

$$\mathbf{px} \leq 0 \quad \text{for } \mathbf{x} \in X.$$

That is, X is below the hyperplane $\mathbf{px} = 0$.

For any **x**, **pz** is bounded from below by **px**. Hence,

$$\mathbf{pz} = \frac{\mathbf{p}(\lambda \mathbf{z})}{\lambda} \geq \frac{\mathbf{px}}{\lambda} \quad \text{for } \lambda > 0.$$

Letting $\lambda \to 0$, we get

$$\mathbf{pz} \geq 0 \quad \text{for } \mathbf{z} \in \mathbb{R}^L_+.$$

That is, $\mathbb{R}^L_+$ is above the hyperplane $\mathbf{px} = 0$.
Suppose $p_l < 0$ for some l. Let $\mathbf{z} \in \mathbb{R}^L_+$ obey $z_l > 0$ and $z_{l'} = 0$ for $l' \neq l$. Then,

$$\mathbf{pz} = \sum_{l'=1}^{L} p_{l'} z_{l'} = \underbrace{p_l}_{<0} \underbrace{z_l}_{>0} < 0,$$

a contradiction. Q.E.D.

3 The 2nd welfare theorem

Theorem (2nd Welfare Theorem with CCMs): *Let the utility functions be strictly increasing and strictly quasi-concave. Suppose $(\mathbf{c}^{i*})_{i=1}^{I}$ is a Pareto-optimal allocation. Then there are a distribution of endowments $(\mathbf{y}^{i})_{i=1}^{I}$ and a price vector $\mathbf{q}$ such that $((\mathbf{c}^{i*})_{i=1}^{I}, \mathbf{q})$ is an ECCM given the endowments $(\mathbf{y}^{i})_{i=1}^{I}$.*

Proof: By the alternative characterization of quasi-concavity of U, the sets of consumption bundles

$$X^{i} = \{\mathbf{c}^{i} | U^{i}(\mathbf{c}^{\mathbf{i}}) \geq U^{i}(\mathbf{c}^{i*})\}$$

preferred to $\mathbf{c}^{i*}$ are convex.

Let X denote the set of aggregate consumption vectors which can be divided across consumers such that each consumer is better-off than with $\mathbf{c}^{i*}$:

$$X = \{\mathbf{c} | \mathbf{c} = \sum_{i=1}^{I} \mathbf{c}^i, \mathbf{c}^i \in X^i\}.$$

Consider $\mathbf{c}', \mathbf{c}'' \in X$. This implies that there exist $\mathbf{c^i}', \mathbf{c^i}'' \in X^i$. Using convexity of X^i,

$$\lambda \mathbf{c}' + (1-\lambda)\mathbf{c}'' = \lambda \sum_{i=1}^{I} \mathbf{c^i}' + (1-\lambda) \sum_{i=1}^{I} \mathbf{c^i}''$$

$$= \sum_{i=1}^{I} \underbrace{[\lambda \mathbf{c^i}' + (1-\lambda)\mathbf{c^i}'']}_{\in X^i} \in X \quad \text{for } \lambda \in [0,1].$$

Hence, X is convex.

Let $\mathbf{a} = \sum_{i=1}^{I} \mathbf{c^{i}}^{*}$. By construction, $\mathbf{a} \in X$. Any $\mathbf{a}'$ obtained by reducing some component of $\mathbf{a}$ means that some consumer is worse-off, so $\mathbf{a}$ is on the boundary of X. By the separating hyperplane theorem 2 there exists $\mathbf{q}$ such that

$$\mathbf{qc} \geq \mathbf{qa} \quad \text{for } \mathbf{c} \in X.$$

Let each consumer i be endowed with $\mathbf{c}^{i*}$, and assume $\mathbf{q}$ is the price vector. To prove the theorem, it suffices to show that $\mathbf{c}^{i*}$ maximizes utility. Suppose not. Then there exist i and $\mathbf{c}^i$ such that

$$U^i(\mathbf{c}^i) = U^i(\mathbf{c}^{i*}),\ \mathbf{q}\mathbf{c}^i < \mathbf{q}\mathbf{c}^{i*}.$$

Since i is no worse-off with $\mathbf{c}^i$, we have $\mathbf{c} = \mathbf{c}^i + \sum_{i'=1,i'\neq i}^{I} \mathbf{c}^{i'*} \in X$. Using $\mathbf{a} = \sum_{i=1}^{I} \mathbf{c}^{\mathbf{i}*}$, it follows from the separating hyperplane theorem that

$$\mathbf{q}\left(\mathbf{c}^i + \sum_{i'=1,i'\neq i}^{I} \mathbf{c}^{i'*}\right) \geq \mathbf{q}\sum_{i=1}^{I} \mathbf{c}^{i*},$$

i.e., $\mathbf{q}\mathbf{c}^i \geq \mathbf{q}\mathbf{c}^{i*}$, a contradiction. Q.E.D.

Literature:

- Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory*, Oxford University Press (1995), Chapter 16.

4 The fundamental theorem of asset pricing

The "equilibrium approach" pursued here implies the existence of state prices $\tilde{p}_{t,s}$, determined by

$$q_t \begin{pmatrix} 1 \\ \tilde{\mathbf{p}} \end{pmatrix} = \mathbf{q}$$

(i.e., $\tilde{p}_{t,s} = q_{t+1,s}/q_t$). We have used the state prices to price assets in an ECFM:

$$p_k = \sum_{s=1}^{S} \tilde{p}_{t,s} a_{sk}.$$

In a sense, this approach, building on our model of complete financial markets, is very special.

- Financial markets might be incomplete.
- The “true” model may deviate from our perfectly competitive setup without market imperfections.
- We might not have a model of the total economy at all.

This section shows that the existence of state prices holds much more generally: all we need is arbitrage-free pricing of the assets.

As a motivation, consider the case of certainty ($S = 1$). A bond with payoff 1 and price p_t is the AS for $s = 1$.
If p_t were non-positive, there would be a "free lunch".
Conversely, if there is no free lunch ($p_t > 0$), then the state (AS) price for $s = 1$ is positive.
That is, arbitrage-freeness requires a positive asset price.

Here is the theorem that states that arbitrage-freeness gurantees the existence of state prices $\tilde{p}_{t,s}$. This theorem is often called the fundamental theorem of asset pricing:

Theorem (The fundamental theorem of asset pricing): *If asset prices* $\mathbf{p} > \mathbf{0}$ *are arbitrage-free, then there exist non-negative state prices* $\tilde{\mathbf{p}} = (\tilde{p}_{t,1}, \ldots, \tilde{p}_{t,S})$ *such that*

$$p_k = \sum_{s=1}^{S} \tilde{p}_{t,s} a_{sk}, \quad k = 1, \ldots, K.$$

Proof: Let B denote the set of all payoff vectors attainable with costless portfolios:

$$B = \{\mathbf{x} \in \mathbb{R}^S | \, \mathbf{Az} = \mathbf{x} \text{ for some } \mathbf{z} \text{ with } \mathbf{pz} = 0\}.$$

Arbitrage-freeness implies $B \cap (\mathbb{R}^S_+ \backslash \{\mathbf{0}\}) = \emptyset$, hence $B \cap \mathbb{R}^S_{++} = \emptyset$.

B is convex. To see this, let $\mathbf{x}', \mathbf{x}'' \in B$. Then there are $\mathbf{z}', \mathbf{z}'' \in \mathbb{R}^S$ such that

$$0 = \lambda \mathbf{p}\mathbf{z}' + (1-\lambda)\mathbf{p}\mathbf{z}'' = \mathbf{p}[\lambda \mathbf{z}' + (1-\lambda)\mathbf{z}'']$$

$$\lambda \mathbf{x}' + (1-\lambda)\mathbf{x}'' = \lambda \mathbf{A}\mathbf{z}' + (1-\lambda)\mathbf{A}\mathbf{z}'' = \mathbf{A}[\lambda \mathbf{z}' + (1-\lambda)\mathbf{z}''].$$

So $\lambda \mathbf{x}' + (1-\lambda)\mathbf{x}'' \in B$, since it is generated by the costless portfolio $\lambda \mathbf{z}' + (1-\lambda)\mathbf{z}''$.

Due to Separating hyperplane theorem 3, there exists $\tilde{\mathbf{p}}' \in \mathbb{R}^S$, $\tilde{\mathbf{p}}' \geq \mathbf{0}$ such that

$$\tilde{\mathbf{p}}'\mathbf{x} \leq 0, \quad \mathbf{x} \in B.$$

More specifically,

$$\tilde{\mathbf{p}}'\mathbf{x} = 0, \quad \mathbf{x} \in B.$$

$\mathbf{x} \in B$ implies $\mathbf{x} = \mathbf{A}\mathbf{z}$ and $\mathbf{p}\mathbf{z} = 0$ for some $\mathbf{z}$, so $-\mathbf{x} = \mathbf{A}(-\mathbf{z})$ and $\mathbf{p}(-\mathbf{z}) = 0$, i.e., $-\mathbf{x} \in B$. If $\tilde{\mathbf{p}}'\mathbf{x} < 0$ for $\mathbf{x} \in B$, then $\tilde{\mathbf{p}}'(-\mathbf{x}) > 0$ and $-\mathbf{x} \in B$, a contradiction.

Since $\tilde{p}'_s \geq 0$ and $a_{sk} \geq 0$, we have

$$\sum_{s=1}^{S} \tilde{p}'_s a_{sk} \geq 0, \quad k = 1, \ldots, K.$$

Assume

$$(a_{s1}, \ldots, a_{sK}) \neq \mathbf{0}, \quad s = 1, \ldots, S.$$

This entails no loss of generality, for if $(a_{s1}, \ldots, a_{sK}) = \mathbf{0}$, then set $\tilde{p}_{t,s} > 0$ arbitrary and drop the state s from the analysis. Since $\tilde{\mathbf{p}}' \neq \mathbf{0}$, there is some state s with $\tilde{p}'_s > 0$. Since some component of $(a_{s1}, \ldots, a_{sK})$ is positive, we have

$$\sum_{s=1}^{S} \tilde{p}'_s a_{sk} > 0 \text{ for some } k.$$

Without loss of generality, let this asset be $k = 1$.

Consider the equation:

$$p_k = \alpha \sum_{s=1}^{S} \tilde{p}'_s a_{sk}.$$

For $k = 1$, $\alpha = p_1/(\sum_{s=1}^{S} \tilde{p}'_s a_{s1})$ (> 0) is well defined. The equation holds true for all $k = 1, \ldots, K$. Suppose not. Without loss of generality, let $k = 2$ be the asset for which the equality does not hold:

$$\frac{\sum_{s=1}^{S} \tilde{p}'_s a_{s1}}{p_1} \neq \frac{\sum_{s=1}^{S} \tilde{p}'_s a_{s2}}{p_2}.$$

Consider the following portfolio:

$$\mathbf{z} = \left(\frac{1}{p_1}, -\frac{1}{p_2}, 0 \ldots, 0\right).$$

The portfolio is costless: $\mathbf{pz} = 0$. So $\mathbf{Az} \in B$ and

$$\tilde{\mathbf{p}}'(\mathbf{Az}) = 0.$$

The payoff generated by the portfolio is

$$\mathbf{Az} = \begin{pmatrix} a_{11}z_1 + a_{12}z_2 \\ \vdots \\ a_{S1}z_1 + a_{S2}z_2 \end{pmatrix}.$$

Therefore,

$$
\begin{aligned}
\tilde{\mathbf{p}}'(\mathbf{Az}) &= \begin{pmatrix} \tilde{p}'_1 \\ \vdots \\ \tilde{p}'_S \end{pmatrix} \begin{pmatrix} a_{11}z_1 + a_{12}z_2 \\ \vdots \\ a_{S1}z_1 + a_{S2}z_2 \end{pmatrix} \\
&= z_1 \sum_{s=1}^{S} \tilde{p}'_s a_{s1} + z_2 \sum_{s=1}^{S} \tilde{p}'_s a_{s2} \\
&= \frac{\sum_{s=1}^{S} \tilde{p}'_s a_{s1}}{p_1} - \frac{\sum_{s=1}^{S} \tilde{p}'_s a_{s2}}{p_2} \\
&\neq 0,
\end{aligned}
$$

a contradiction.
Define $\tilde{\mathbf{p}} = \alpha \tilde{\mathbf{p}}'$. This completes the proof of the theorem.
Q.E.D.

Literature:

- Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory*, Oxford University Press (1995), Section 19.E.
- Magill, Michael, and Martine Quinzii, *Theory of Incomplete Markets*, MIT (2002), Section 9.

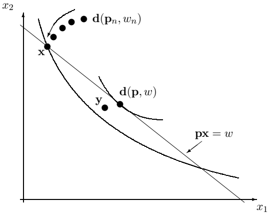
5 Math III: Math for the existence problem

Theorem (Theorem of the maximum): *Let* $\mathbf{p} \in \mathbb{R}^L_{++}$, $w \geq 0$, $X = \{\mathbf{x} | \mathbf{x} \in \mathbb{R}^L_+, \mathbf{px} \leq w\}$, *and* $f : \mathbb{R}^L_+ \to \mathbb{R}$ *continuous, strictly increasing, and stricty quasi-concave. Then the solution* $\mathbf{d}(\mathbf{p}, w) : \mathbb{R}^L_{++} \times \mathbb{R}_+ \to \mathbb{R}^L_+$ *to*

$$\max_{\mathbf{x} \in X} : f(\mathbf{x})$$

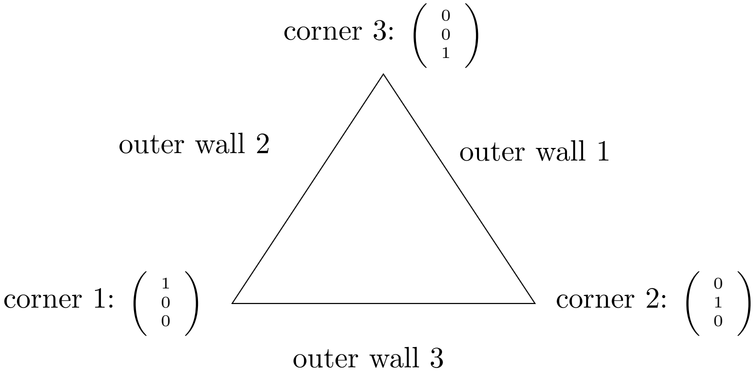
is a continuous function.

Proof: The set of maximizers **d** is non-empty due to the Weierstrass theorem.
d is a function (i.e., single-valued). If there were two different maximizers **x** and **y**, then by quasi-concavity, $f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) > f(\mathbf{x}) = f(\mathbf{y})$, a contradiction.

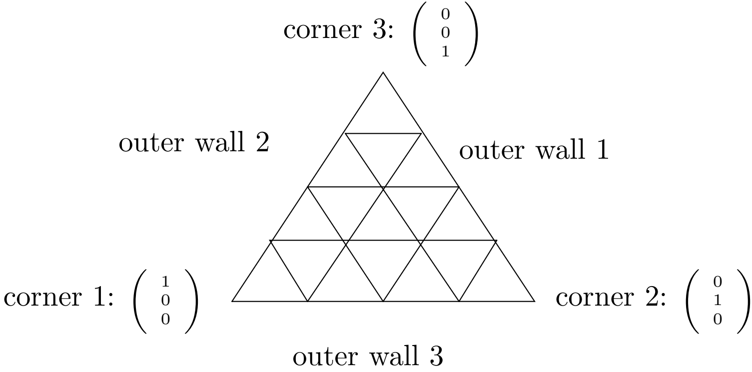


d is continuous. Suppose not. Consider a sequence $(\mathbf{p}_n, w_n) \to (\mathbf{p}, w)$. Suppose $\mathbf{d}(\mathbf{p}_n, w_n) \to \mathbf{x} \neq \mathbf{d}(\mathbf{p}, w)$. $\mathbf{p}_n\mathbf{d}(\mathbf{p}_n, w_n) \leq w_n$ implies $\mathbf{px} \leq w$. Since $\mathbf{d}(\mathbf{p}, w) \neq \mathbf{x}$ is optimal given $(\mathbf{p}, w)$, we must have $f(\mathbf{d}(\mathbf{p}, w)) > f(\mathbf{x})$. As f is continuous and strictly increasing, there is $\mathbf{y}$ close to $\mathbf{d}(\mathbf{p}, w)$ such that $\mathbf{py} < w$ and $f(\mathbf{y}) > f(\mathbf{x})$. The former inequality implies $\mathbf{p}_n\mathbf{y} < w_n$ for n large enough. From the fact that $\mathbf{d}(\mathbf{p}_n, w_n)$ is optimal given $(\mathbf{p}_n, w_n)$, it follows that $f(\mathbf{d}(\mathbf{p}_n, w_n)) > f(\mathbf{y})$. By continuity of f, $f(\mathbf{x}) > f(\mathbf{y})$, a contradiction. Q.E.D.

Consider the triangle formed by the three vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ as a (triangular) house. The three vertices are called corners 1, 2, and 3, respectively. The side of the triangle opposite to corner *i* is called outer wall *i*.

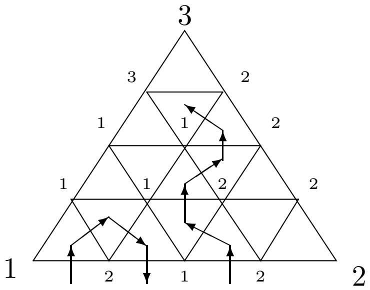


The house is divided into triangular rooms by walls. Any two rooms either share one common wall or none at all.



Give each corner of each room a number, 1, 2, or 3. The only labelling rules are:

- Corners 1, 2, 3 are labeled 1, 2, 3 respectively;
- No corner on outer wall i is labeled i ($i = 1, 2, 3$).



Theorem (Sperner's Lemma): *There exists an odd number of rooms whose corners are labeled 1, 2, 3.*

Proof: Say a wall has a door if its corners are labeled 1 and 2. Start at corner 1 and go along outer wall 3 until you reach a door. Enter the room.

Walk the house according to the following rules:

- If the room you enter is labeled 1, 2, 3: stop.
- If it's labeled 1, 2, 1 or 1, 2, 2, there is a second door besides the one through which you entered: go through this door.

The walk obeys the following two observations:

- You never enter the same room twice.
- The walk ends in a room labeled 1, 2, 3 or you exit the house through a door in wall 3.

Since the number of doors in wall 3 is odd, there is an odd number of walks which terminate in a room labeled 1, 2, 3. This proves that there is a room labeled 1, 2, 3.
If, in addition, there is a room no walk starting at wall 3 leads into, then start another walk obeying the two rules there. By virtue of the two observations made, the walk ends in another such room. So this type of rooms come in pairs.
This proves that the number of rooms labeled 1, 2, 3 is odd.

The theorem easily generalizes to higher dimensions by means of induction on the number of dimensions.
Suppose the assertion of the theorem holds true for an $n-1$-dimensional simplex, spanned by n points. Consider the n-dimensional simplex made up of $n+1$ points $(1,0,\ldots,0),\ldots,(0,0,\ldots,1)$.
By the induction hypothesis, the $n-1$-dimensional subsimplex obtained by deleting the $n+1$st point has an odd number of $n-1$-dimensional rooms labeled $(1,0,\ldots,0),\ldots,(0,0,\ldots,1)$.
Walking the house analogously to the rules above proves that there is an odd number of n-dimensional rooms.

Theorem (Brouwer's fixed point theorem): *Let* $\mathbf{f} : \Delta \to \Delta$ *be a continuous mapping from the two-dimensional simplex on itself. Then there exists a fixed point* $\mathbf{x} \in \Delta$ *such that* $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.

Proof: Consider an arbitrarily fine triangularization of the house into rooms.
Suppose no corner is a fixed point of **f** since otherwise the assertion of the theorem is trivially true.

Consider the following labelling procedure:

- Corners 1, 2, 3 are labeled 1, 2, 3 respectively;
- To each other point, attach a label i ($\in \{1, 2, 3\}$) such that $f_i(\mathbf{x}) < x_i$.

$f_i(\mathbf{x}) < x_i$ for some i, since $\sum_i x_i = \sum_i f_i(\mathbf{x}) = 1$ and $x_i \leq f_i(\mathbf{x})$ for all i imply that $x_i = f_i(\mathbf{x})$ for all i. So the labelling procedure is feasible.
Points on outer wall i satisfy $x_i = 0$. So $f_i(\mathbf{x}) \geq x_i$, and the points are not labeled i. That is, the labelling procedure conforms to the rule specified above.
By Sperner's lemma, there is a room labeled 1, 2, 3.
A room labeled 1, 2, 3 satisfies $f_i(\mathbf{x}) < x_i$ for $i \in \{1, 2, 3\}$.

Consider a sequence of ever finer triangularizations of the house into rooms. For each triangularization, there is a room such that $f_i(\mathbf{x}) < x_i$ for $i \in \{1, 2, 3\}$.
Since $\mathbf{x}$ is in the compact set Δ, we can pick a convergent subsequence.
In the limit, we must have $f_i(\mathbf{x}) = x_i$ for $i \in \{1, 2, 3\}$. Q.E.D.

This proof generalizes immediately to higher dimensions.

8 Existence

Theorem (Existence): *Let the utility functions be strictly increasing, strictly quasi-concave, and continuous. Then an equilibrium exists.*

Proof: Let the solution $(c_t^i, c_{t+1,1}^i, c_{t+1,2}^i)$ to the consumers' utility maximization problem problem be denoted as $(c_0(\mathbf{q}), c_1(\mathbf{q}), c_2(\mathbf{q})) = \mathbf{c}^i(\mathbf{q})$. The Theorem of the maximum ensures that these are continuous functions.

Let $\mathbf{z}(\mathbf{q})$ denote the excess demand function:

$$\mathbf{z}(\mathbf{q}) = \sum_{i=1}^{I} [\mathbf{c}^i(\mathbf{q}) - \mathbf{y}^i].$$

The excess demand functions are continuous.
From the budget constraints, $\mathbf{q}\mathbf{c}^i(\mathbf{q}) = \mathbf{q}\mathbf{y}^i$ for all $\mathbf{q}$ and all i.
Hence,

$$\mathbf{q}\mathbf{z}(\mathbf{q}) = \sum_{i=1}^{I} \mathbf{q}[\mathbf{c}^i(\mathbf{q}) - \mathbf{y}^i] = 0.$$

This relation is called ***Walras' law***: the value of the excess demands is zero.

In a slight abuse of notation, let
$\mathbf{q} = (q_t, q_{t+1,1}, q_{t+1,2}) = (q_0, q_1, q_2)$.
Consider the Gale-Nikaido mapping $\mathbf{f} = (f_0, f_1, f_2)$ defined by:

$$f_j(\mathbf{q}) = \frac{q_j + \max\{z_j(\mathbf{q}), 0\}}{1 + \sum_{j'=0}^{2} \max\{z_{j'}(\mathbf{q}), 0\}}, \quad j = 0, 1, 2.$$

Since the model is homogeneous of degree zero in $\mathbf{q}$, we are free to consider price vectors $\mathbf{q}$ in the two-dimensional unit simplex $\Delta \equiv \{\mathbf{q} | \sum_{j=0}^{2} q_j = 1, q_j \geq 0, j = 0, 1, 2\}$.

f is a continuous mapping from the unit simplex on itself. By Brouwer's theorem, there is a fixed point, such that $\mathbf{q} = \mathbf{f}(\mathbf{q})$. Using Walras' law, it follows that the fixed point is an equilibrium price vector.
So there is **q** such that

$$q_j = \frac{q_j + \max\{z_j(\mathbf{q}), 0\}}{1 + \sum_{j'=0}^{2} \max\{z_{j'}(\mathbf{q}), 0\}}, \quad j = 0, 1, 2.$$

Multiply by $z_j(\mathbf{q})$:

$$q_j z_j(\mathbf{q}) = \frac{q_j z_j(\mathbf{q}) + z_j(\mathbf{q}) \max\{z_j(\mathbf{q}), 0\}}{1 + \sum_{j'=0}^{2} \max\{z_{j'}(\mathbf{q}), 0\}}, \quad j = 0, 1, 2.$$

Sum over all j:

$$\underbrace{\sum_{j=0}^{2} q_j z_j(\mathbf{q})}_{=0} = \frac{\overbrace{\sum_{j=0}^{2} q_j z_j(\mathbf{q})}^{=0} + \sum_{j=0}^{2} z_j(\mathbf{q}) \max\{z_j(\mathbf{q}), 0\}}{1 + \sum_{j'=0}^{2} \max\{z_{j'}(\mathbf{q}), 0\}}.$$

Use Walras' law:

$$0 = \frac{\sum_{j=0}^{2} z_j(\mathbf{q}) \max\{z_j(\mathbf{q}), 0\}}{1 + \sum_{j'=0}^{2} \max\{z_{j'}(\mathbf{q}), 0\}}.$$

Each term in the sum in the numerator is non-negative, so each term has to be equal to zero:

$$z_j(\mathbf{q}) = 0, \quad j = 0, 1, 2.$$

There's one subtlety we've ignored so far. We've presupposed that a solution to the consumers' utility maximization problem exists. According to the Weierstrass theorem, this requires that the domain of the problem is compact. However, the unit simplex includes price vectors with zero components, and when one price is zero, the individuals' budget sets are not compact. This problem is handled as follows.

Add the restriction $\mathbf{c}^i \leq 2\sum_{i'=1}^{I} \mathbf{y}^{i'}$ to i's problem. This makes the domain compact end ensures the existence of a solution $\mathbf{c}^i(\mathbf{q})$.

So there is a price vector $\mathbf{q}$ which clears the markets given the "demand functions" $\mathbf{c}^i(\mathbf{q})$. By construction, $\mathbf{c}^i(\mathbf{q})$ is in the interior of the set of consumption bundles which satisfy $\mathbf{c}^i \leq 2\sum_{i'=1}^{I} \mathbf{y}^{i'}$.

The restriction of the domain is immaterial. Suppose there's a consumption vector $\mathbf{c}'^i$ in i's budget set and which i prefers to $\mathbf{c}^i(\mathbf{q})$ but which does not satisfy $\mathbf{c}^i \leq 2\sum_{i'=1}^{I} \mathbf{y}^{i'}$.

Since both $\mathbf{c}^i$ and $\mathbf{c}'^i$ are in i's budget set, so is a mixture of them.

- Because of strict quasi-concavity, as $\mathbf{c}'^i$ is preferred to $\mathbf{c}^i$, so is any mixture of the two.
- As $\mathbf{c}^i(\mathbf{q})$ is in the interior of the set of consumption bundles which satisfy $\mathbf{c}^i \leq 2\sum_{i'=1}^{I} \mathbf{y}^{i'}$, so is a mixture with a sufficiently high weight on $\mathbf{c}^i$.

It follows that there is a mixture of $\mathbf{c}^i$ and $\mathbf{c}'^i$ with a sufficiently high weight on the former which is in i's budget set, is preferred to $\mathbf{c}^i$, and satisfies $\mathbf{c}^i \leq 2\sum_{i'=1}^{I} \mathbf{y}^{i'}$, a contradiction. Q.E.D.

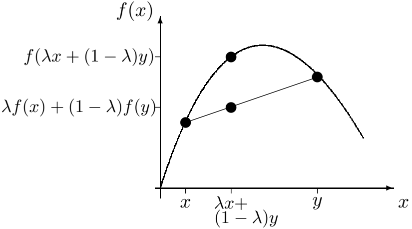
Literature:

- Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory*, Oxford University Press (1995), Sections 17.A-C.
- Nikaido, Hukukane (1970), *Introduction to Sets and Mappings in Modern Economics*, Amsterdam: North-Holland, Chapters 8, 10.

7 Math IV: Concave optimization

Definition: Let $X \subset \mathbb{R}^L$. A function $f : X \to \mathbb{R}$ is ***concave*** if for any $\mathbf{x}, \mathbf{y} \in X$,

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \quad \text{for } \lambda \in [0,1].$$



Theorem (Kuhn-Tucker theorem): Let $X \subset \mathbb{R}^L$ be convex and $f : X \to \mathbb{R}$ and $g_i : X \to \mathbb{R}$, $i = 1, \ldots, I$ concave functions. Suppose there is $\mathbf{c} \in X$ such that $g_i(\mathbf{c}) > 0$, $i = 1, \ldots, I$. Suppose $\mathbf{x}^*$ solves

$$\max_{\mathbf{x} \in X} : f(\mathbf{x})$$

$$\text{s.t.: } g_i(\mathbf{x}) \geq 0 \quad \text{for } i = 1, \ldots, I.$$

Let

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{I} \lambda_i g_i(\mathbf{x}).$$

Then there is $\boldsymbol{\lambda}^* \in \mathbb{R}^I_+$ such that $\mathbf{x}^*$ maximizes $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}^*)$ and $\lambda_i^* g_i(\mathbf{x}^*) = 0$, $i = 1, \ldots, I$.

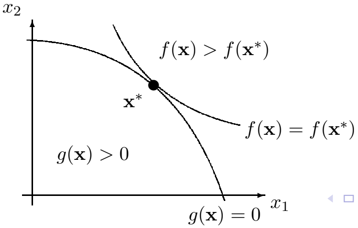
Proof: Define $g_0(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}^*)$ and $\mathbf{g} = (g_0, g_1, \ldots, g_I) : X \to \mathbb{R}^{I+1}$. Consider the set

$$Y = \{\mathbf{y} | \, \mathbf{y} \in \mathbb{R}^{I+1}, \mathbf{y} \leq \mathbf{g}(\mathbf{x}) \text{ for some } \mathbf{x} \in X\}.$$

Suppose $\mathbf{y} \in Y$, $\mathbf{y} \in \mathbb{R}^{I+1}_{++}$. That is, the constraints are not binding, and the value of the objective function is greater than maximal, a contradiction. Formally, $0 < y_i \leq g_i(\mathbf{x})$, $i = 1, \ldots, I$, implies

$$y_0 = g_0(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}^*) \leq 0.$$

So Y and the interior of $\mathbb{R}^{I+1}_{+}$ are disjoint.



Let $\mathbf{y}', \mathbf{y}'' \in Y$. There are $\mathbf{x}', \mathbf{x}'' \in X$ such that $\mathbf{y}' \leq \mathbf{g}(\mathbf{x}')$ and $\mathbf{y}'' \leq \mathbf{g}(\mathbf{x}'')$. Because of concavity of $\mathbf{g}$,

$$\mathbf{g}(\lambda \mathbf{x}' + (1-\lambda)\mathbf{x}'') \geq \lambda \mathbf{g}(\mathbf{x}') + (1-\lambda)\mathbf{g}(\mathbf{x}'') = \lambda \mathbf{y}' + (1-\lambda)\mathbf{y}''.$$

Because of convexity of X, $\lambda \mathbf{x}' + (1-\lambda)\mathbf{x}'' \in X$. So $\lambda \mathbf{y}' + (1-\lambda)\mathbf{y}'' \in Y$, i.e., Y is convex.
Separating hyperplane theorem 3 implies that there exists a hyperplane $\mathbf{px} = 0$ with $\mathbf{p} \geq \mathbf{0}$ such that $\mathbf{py} \leq 0$ for $\mathbf{y} \in Y$.

Let

$$\mathbf{g}(X) = \{\mathbf{y} | \, \mathbf{y} \in \mathbb{R}^{l+1}, \mathbf{y} = \mathbf{g}(\mathbf{x}) \text{ for some } \mathbf{x} \in X\}.$$

Clearly, $g(X) \subset Y$. So $\mathbf{py} \leq 0$ for $\mathbf{y} \in \mathbf{g}(X)$, i.e.,

$$\mathbf{pg}(\mathbf{x}) = p_0[f(\mathbf{x}) - f(\mathbf{x}^*)] + \sum_{i=1}^{l} p_i g_i(\mathbf{x}) \leq 0 \text{ for } \mathbf{x} \in X.$$

For $\mathbf{x} = \mathbf{x}^*$, we have $\sum_{i=1}^{l} p_i g_i(\mathbf{x}^*) \leq 0$. As $p_i \geq 0$ and $g_i(\mathbf{x}^*) \leq 0$, this implies

$$p_i g_i(\mathbf{x}^*) = 0 \quad \text{for } i = 1, \ldots, l.$$

Suppose $p_0 = 0$. Then $\sum_{i=1}^{l} p_i g_i(\mathbf{x}) \leq 0$ for $\mathbf{x} \in X$. Since $\sum_{i=1}^{l} p_i g_i(\mathbf{c}) \geq 0$, we have $\sum_{i=1}^{l} p_i g_i(\mathbf{c}) = 0$. $g_i(\mathbf{c}) > 0$ for $i = 1, \ldots, l$ implies $p_i = 0$ for $i = 1, \ldots, l$. Hence, $\mathbf{p} = \mathbf{0}$. This contradicts the implication of the separating hyperplane theorem. So $p_0 > 0$.

Define

$$\lambda_i = \frac{p_i}{p_0} \text{ for } i = 1, \ldots, l.$$

Then $p_i g_i(\mathbf{x}^*) = 0$ becomes

$$\lambda_i g_i(\mathbf{x}^*) = 0 \quad \text{for } i = 1, \ldots, l.$$

And

$$f(x) + \sum_{i=1}^{l} \lambda_i g_i(\mathbf{x}) \leq f(\mathbf{x}^*) \leq f(\mathbf{x}^*) + \sum_{i=1}^{l} \lambda_i g_i(\mathbf{x}^*).$$

Q.E.D.

Definition: Let $X \subset \mathbb{R}$. $f : X \to \mathbb{R}$ is differentiable at $x \in X$ if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. The derivative is then denoted $f'(x)$ or $df(x)/dx$. If f is differentiable at all x in the interior of X, then it is ***differentiable***.
Definition: Let $X \subset \mathbb{R}^L$. $f : X \to \mathbb{R}$ is partially differentiable with respect to x_l at $\mathbf{x} \in X$ if

$$\lim_{h \to 0} \frac{f(x_1, \ldots, x_l + h, \ldots, x_L) - f(\mathbf{x})}{h}$$

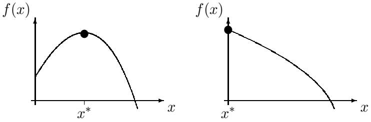
exists. The partial derivative is then denoted $\partial f(\mathbf{x})/\partial x_l$. If the partial derivatives exist at all $\mathbf{x}$ in the interior of X, then f is ***differentiable***.

Theorem (Necessary conditions for unconstrained maximization): Let $f : \mathbb{R}^L_+ \to \mathbb{R}$ be continuously differentiable. If $\mathbf{x}^*$ solves

$$\max_{\mathbf{x}\in\mathbb{R}^L_+} : f(\mathbf{x}),$$

then

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_l} \leq 0, \ \ \frac{\partial f(\mathbf{x}^*)}{\partial x_l} x_l^* = 0 \quad \text{for } l = 1, \ldots, L.$$



Proof: For any $\mathbf{x} \in \mathbb{R}^L_+$, let $\mathbf{x}(\lambda) = \lambda \mathbf{x} + (1 - \lambda)\mathbf{x}^*$. Since $\mathbf{x}^* = \mathbf{x}(0)$ maximizes f,

$$\frac{f(\mathbf{x}(\lambda)) - f(\mathbf{x}(0))}{\lambda} \leq 0 \quad \text{for } \mathbf{x} \in \mathbb{R}^L_+, \lambda \in (0, 1].$$

Taking the limit $\lambda \to 0$,

$$\frac{df(\mathbf{x}(0))}{d\lambda} \leq 0 \quad \text{for } \mathbf{x} \in \mathbb{R}^L_+.$$

Using $\mathbf{x}(0) = \mathbf{x}^*$ and

$$\frac{df(\mathbf{x}(\lambda))}{d\lambda} = \sum_{l=1}^{L} \frac{\partial f(\mathbf{x}(\lambda))}{\partial x_l}(x_l - x_l^*),$$

this becomes

$$\sum_{l=1}^{L} \frac{\partial f(\mathbf{x}^*)}{\partial x_l} x_l \leq \sum_{l=1}^{L} \frac{\partial f(\mathbf{x}^*)}{\partial x_l} x_l^* \quad \text{for } \mathbf{x} \in \mathbb{R}^L_+.$$

First, choose $\mathbf{x} \in \mathbb{R}^L_+$ such that $x_l = \mu > 0$ and $x_{l'} = 0$ for $l' \neq l$. Then,

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_l} \leq \frac{\frac{\partial f(\mathbf{x}^*)}{\partial x_l}}{\mu} \quad \text{for } \mu > 0.$$

Letting $\mu \to 0$ proves the first assertion of the theorem.
Second, choose $\mathbf{x} = \mathbf{0} \in \mathbb{R}^L_+$. This yields

$$\sum_{l=1}^{L} \underbrace{\frac{\partial f(\mathbf{x}^*)}{\partial x_l}}_{\leq 0} \underbrace{x_l^*}_{\geq 0} \geq 0.$$

Since $\partial f(\mathbf{x}^*)/\partial x_l \leq 0$ and $x_l^* \geq 0$, each term in the sum must be equal to zero. This implies the validity of the second assertion of the theorem. Q.E.D.

Theorem (Constrained maximization): Let $X = \mathbb{R}^L$ and f and g_i, $i = 1, \ldots, I$ be continuously differentiable in the Kuhn-Tucker theorem. Then there is $\boldsymbol{\lambda}^* \in \mathbb{R}^I_+$ such that

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_l} \leq 0 \ , \frac{\partial \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_l} x_l^* = 0 \quad \text{for } l = 1, \ldots, L,$$

and

$$\lambda_i^* g_i(\mathbf{x}^*) = 0, i = 1, \ldots, I.$$

Proof: This restates the condition that $\mathbf{x}^*$ maximizes $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}^*)$ using the theorem on the necessary conditions for unconstrained maximization. Q.E.D.

Theorem (Envelope theorem): Let $f(\mathbf{x}, c)$, $g_i(\mathbf{x}, c)$, $i = 1, \ldots, I$, and, hence, the Langrangean $\mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, c)$ depend on a parameter $c \in \mathbb{R}$ in the differentiable constrained maximization problem. Suppose an interior solution $\mathbf{x}^* > \mathbf{0}$ obtains. Then

$$\frac{df(\mathbf{x}^*, c)}{dc} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, c)}{\partial c}.$$

In the unconstrained case, this boils down to

$$\frac{df(\mathbf{x}^*, c)}{dc} = \frac{\partial f(\mathbf{x}^*, c)}{\partial c}.$$

Proof: $\mathbf{x}^* > \mathbf{0}$ implies

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, c)}{\partial x_l} = \frac{\partial f(\mathbf{x}^*, c)}{\partial x_l} + \sum_{i=1}^{I} \lambda_i^* \frac{\partial g_i(\mathbf{x}^*, c)}{\partial x_l} = 0.$$

Differentiating $f(\mathbf{x}^*, c)$ gives

$$\frac{df(\mathbf{x}^*, c)}{dc} = \frac{\partial f(\mathbf{x}^*, c)}{\partial c} + \sum_{l=1}^{L} \frac{\partial f(\mathbf{x}^*, c)}{\partial x_l} \frac{dx_l^*}{dc}$$

or

$$\frac{df(\mathbf{x}^*, c)}{dc} = \frac{\partial f(\mathbf{x}^*, c)}{\partial c} - \sum_{i=1}^{I} \lambda_i^* \sum_{l=1}^{L} \frac{\partial g_i(\mathbf{x}^*, c)}{\partial x_l} \frac{dx_l^*}{dc}.$$

Differentiating the i-th constraint gives

$$\frac{\partial g_i(\mathbf{x}^*, c)}{\partial c} + \sum_{l=1}^{L} \frac{\partial g_i(\mathbf{x}^*, c)}{\partial x_l} \frac{dx_l^*}{dc} = 0 \quad \text{for } i = 1, \ldots, I.$$

So

$$\begin{aligned} \frac{df(\mathbf{x}^*, c)}{dc} &= \frac{\partial f(\mathbf{x}^*, c)}{\partial c} + \sum_{i=1}^{I} \frac{\partial g_i(\mathbf{x}^*, c)}{\partial c} \\ &= \frac{\partial \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, c)}{\partial c}. \end{aligned}$$

Q.E.D.