Connexive logic:
X-Phi results and coherence-based probability semantics

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A basic, connexive intuition

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T	T	T	**T**	T
F	T	F	**T**	T
T	F	T	**F**	F
F	F	F	**T**	F

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- $A \supset \neg A$ is equivalent to $\neg A$.

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- $A \supset \neg A$ is equivalent to $\neg A$.

A	A	$\supset$	$\neg$	A
T	T	**F**	F	T
F	F	**T**	T	F

- $\neg A \supset A$ is equivalent to A.

A	$\neg$	A	$\supset$	A
T	F	T	**T**	T
F	T	F	**F**	F

Some connexive principles

- **Aristotle's thesis:** $\sim(\sim A \rightarrow A)$

A	$\neg$	$(\neg$	A	$\supset$	$A)$
T	**F**	F	T	T	T
F	**T**	T	F	F	F

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F	**T**	T	F	F	F

- **Abelard's first principle:** $\sim((A \to B) \wedge (A \to \sim B))$

A	B	$\neg$	$((A$	$\supset$	$B)$	$\wedge$	$(A$	$\supset$	$\sim$	$B))$
T	T	**T**	T	T	T	F	T	F	F	T
F	T	**F**	F	T	T	T	F	T	F	T
T	F	**T**	T	F	F	F	T	T	T	F
F	F	**F**	F	T	F	T	F	T	T	F

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- **Abelard's first principle:** $\sim((A \to B) \wedge (A \to \sim B))$

A	B	$\neg$	$((A$	$\supset$	$B)$	$\wedge$	$(A$	$\supset$	$\sim$	$B))$
T	T	**T**	T	T	T	F	T	F	F	T
F	T	**F**	F	T	T	T	F	T	F	T
T	F	**T**	T	F	F	F	T	T	T	F
F	F	**F**	F	T	F	T	F	T	T	F

- **Boethius' thesis:** $(A \to B) \to \sim(A \to \sim B)$

A	B	$(A$	$\to$	$B)$	$\supset$	$\neg$	$(A$	$\supset$	$\neg$	$B)$
T	T	T	T	T	**T**	T	T	F	F	T
F	T	F	T	T	**F**	F	F	T	F	T
T	F	T	F	F	**T**	F	T	T	T	F
F	F	F	T	F	**F**	F	F	T	T	F

Previous data

Table: Collection of data from previously published experiments in %. As all of the studies differ slightly in their exact phrasing and presentation of the material, we opted to use pre-theoretic symbols +, -, ? to categorize their results.

	Pfeifer & Tulkki 2017			Pfeifer 2012			McCull 2012			Pfeifer & Stöckle-Schobel 2015			Pfeifer & Yama 2017		
	n = 60			n = 40			n = 89			n = 40			n = 63		
Formula name	+	-	?	+	-	?	+	-	?	+	-	?	+	-	?
Negated Identity	12	75	13	10	88	2		NA		10	78	13	6	63	30
Conjunction Elimination		NA			NA		78	20	2		NA			NA	
Contingent Conditional		NA		0	13	88		NA			NA			NA	
Identity		NA		93	3	5	97	3	0		NA			NA	
Arbitrary Fallacy		NA			NA		6	88	7		NA			NA	
Aristotle's Thesis'	77	7	17	78	18	5	88	7	6	68	23	10	76	11	13
Aristotle's Thesis	72	12	17	80	13	8		NA		70	20	10	65	16	19
Boethius' Thesis		NA			NA		84	8	8		NA			NA	

Overview

- Connexive logics generally are constructed to conform to human intuition about connexive principles.
- Coherence-based probability logic features conditional events $(C|A)$ that are true if both A and C hold, false if A holds but C does not, and *void* otherwise.
- All conditioning events are taken to be possible ($\neq \emptyset$).
- Probability are interpreted as subjective degrees of belief, and probability assignments are coherent if they avoid Dutch books.
- Imagine your degree of belief in a conditional as the amount you are willing to pay for a bet that it holds (payoff of 1), and if it is void you get your money back.

Approach 1: Non-iterated conditionals

(see Pfeifer & Sanfilippo 2021)

From the conditional we infer a probabilistic constraint as follows:

- For $A \rightarrow C, p(C|A) = 1$
- For $A \rightarrow {\sim}C, p(C|A) = 0$
- For ${\sim}(A \rightarrow C), p(C|A) < 1$
- For ${\sim}(A \rightarrow {\sim}C), p(C|A) > 0$

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Definition: A (non-iterated) connexive principle is valid iff the probabilistic constraint associated with the connexive principle is satisfied by every coherent assessment on the involved conditional events.

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Definition: A (non-iterated) connexive principle is valid iff the probabilistic constraint associated with the connexive principle is satisfied by every coherent assessment on the involved conditional events.

E.g., Aristotle's thesis: ${\sim}({\sim}A \to A)$ is associated with the probability constraint $p(A|{\sim}A) < 1$. The only coherent probability assignment for this is $p(A|{\sim}A) = 0$. Hence, as every coherent probability assignment fulfils the probabilistic constraint, Aristotle's Thesis is valid.

Approach 1: Iterated conditionals

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Definition: An iterated connexive principle $\circ \Rightarrow \Box$ is valid iff the probabilistic constraint in the conclusion $\Box$ is satisfied by every coherent extension from the premise $\circ$ to the conclusion $\Box$.

Approach 1: Iterated conditionals

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From the conditional we infer a probabilistic constraint as follows:

- For $A \to C$, $p(C|A) = 1$
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Definition: An iterated connexive principle $\circ \Rightarrow \square$ is valid iff the probabilistic constraint in the conclusion $\square$ is satisfied by every coherent extension from the premise $\circ$ to the conclusion $\square$.

E.g., Boethius' Thesis: $(A \to B) \Rightarrow {\sim}(A \to {\sim}B)$ is associated with the probabilistic constraints $p(B|A) = 1$ (antecedent) and $p(B|A) > 0$ (consequent). If a probability assessment satisfies the probabilistic constraint $p(B|A) = 1$, it also satisfies $p(B|A) > 0$, hence Boethius' Thesis is valid.

Approach 2

(see Pfeifer & Sanfilippo 2021)

- Conditionals $A \to C$ are interpreted as the conditional event $C|A \in \{1, 0, p(C|A)\}$.
- Negated conditionals are interpreted by inner negations, such that $\sim(A \to C)$ is interpreted as $\sim C|A$ (And $\sim(A \to \sim C)$ just as $C|A$).
- Iterated conditionals like $(A \to B) \to (C \to D)$ are usually treated as iterated conditional events $(D|C)|(B|A)$. For non-trivial cases, we use the theory of logical operations among conditional events (see e.g., Gilio & Sanfilippo 2014).

Definition: A connexive principle is valid iff the associated conditional random quantity is constant and equal to one.

E.g., Aristotle's Thesis: $\sim(\sim A \to A)$ is interpreted by (the inner-negated conditional event) $\sim A|\sim A$, which is (by coherence) constant and equal to 1. Hence, Aristotle's Thesis is valid in Approach 2.

Predictions (1/3)

Table: Formulas and connexive principles investigated in both Experiments. Response-predictions according to classical logic (CL), Approach 1 and Approach 2 as to whether a sentence holds (*h*), doesn't hold (*dh*) or one can't tell (*ct*).

Name	Formula	CL	Ap. 1	Ap. 2
Introductory examples				
Excluded Middle	$A \vee \sim A$	h	h	h
Contradiction	$A \wedge \sim A$	dh	dh	dh
Contingent Conjunction	$A \wedge B$	ct	ct	ct
Block 1: *Basic principles*				
Negated Identity	$\sim(A \rightarrow A)$	dh	ct	dh
Conjunction Elimination	$(A \wedge B) \rightarrow A$	h	h	h
Contingent Conditional	$A \rightarrow B$	ct	ct	ct
Self-negated Conditional	$A \rightarrow \sim A$	ct	dh	dh
Identity	$A \rightarrow A$	h	h	h
Arbitrary Fallacy	$A \rightarrow (A \wedge B)$	ct	ct	ct
Aristotle's Thesis′	$\sim(A \rightarrow \sim A)$	ct	h	h
Aristotle's Thesis	$\sim(\sim A \rightarrow A)$	ct	h	h

Predictions (2/3)

Name	Formula	CL	Ap. 1	Ap. 2
Block 2: *Conjunctive principles*				
Negated Abelard's First Principle	$(A \to B) \wedge (A \to \sim B)$	ct	ct	dh
Contingent Conditionals	$(A \to B) \wedge (A \to B)$	ct	ct	ct
Abelard's First Principle	$\sim((A \to B) \wedge (A \to \sim B))$	ct	h	h
Aristotle's Second Thesis	$\sim((A \to B) \wedge (\sim A \to B))$	ct	ct	ct
Contradicting Conditionals	$(A \to B) \wedge \sim(A \to B)$	dh	dh	dh
Block 3: *Iterated principles I*				
Iterated Self-negated Conditional	$(A \to B) \to \sim(A \to B)$	ct	dh	dh
Boethius' Thesis	$(A \to B) \to \sim(A \to \sim B)$	ct	h	h
Iterated Aristotle's Thesis	$\sim(\sim(A \to B) \to (A \to B))$	ct	h	h
Iterated Identity	$(A \to B) \to (A \to B)$	h	h	h
Reversed Boethius' Thesis	$\sim(A \to \sim B) \to (A \to B)$	h	ct	h
Boethius Variation 3	$(A \to B) \to \sim(\sim A \to B)$	ct	ct	ct
Improper Transposition (1/2)	$(A \to B) \to (\sim A \to \sim B)$	ct	ct	ct

Predictions (3/3)

Name	Formula	CL	Ap. 1	Ap. 2
Block 4: *Iterated principles II*				
Iterated Aristotle's Thesis′	$\sim((A \to B) \to \sim(A \to B))$	ct	h	h
Improper Transposition (2/2)	$(A \to B) \to (\sim A \to \sim B)$	ct	ct	ct
Denying a Conjunct	$\sim(A \wedge B) \to (\sim A \to B)$	ct	ct	ct
Boethius' Thesis′	$(A \to \sim B) \to \sim(A \to B)$	ct	h	h
Reversed Boethius' Thesis′	$\sim(A \to B) \to (A \to \sim B)$	h	ct	h
Symmetry	$(A \to B) \to (B \to A)$	ct	ct	ct
Boethius Variation 4	$(\sim A \to B) \to \sim(A \to B)$	ct	ct	ct

Participants

Online questionnaire hosted by `soscisurvey.de`, filled-in in class.

Experiment 1:

- 26 students of the Universities of Vienna and Regensburg.
- *Introductory examples* + tasks from *Block 1: Basic principles* and *Block 2: Conjunctive principles* ($n_1 = 26$).

Experiment 2:

- 46 students of the Universities of Regensburg and Münster.
- *Introductory examples* + tasks from *Block 1: Basic principles* ($n_2 + n_3 = 46$).
- Randomly split between *Block 3: Iterated principles I* ($n_2 = 21$) and *Block 4: Iterated principles II* ($n_3 = 25$).

Method (1/3)

Vignette story: Ida works at a machine which produces playing blocks. Each of these blocks has a shape (*cylinder, cube, ball*) and a size (*small, large*), and the machine to produce blocks in all combinations of these shapes and sizes. **(see e.g., Pfeifer & Tulkki 2017)**

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Ida is waiting in front of the machine and considers the following sentence:
(C) **If** the next playing block is *small*, **then** it is **not** *small*.

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Sample Task: Self-negated conditional, $A \rightarrow {\sim}A$

Method (2/3)

Ida is waiting in front of the machine and considers the following sentence:
(C) **If** the next playing block is *small*, **then** it is **not** *small*.

From here, we first asked participants:

Can Ida even know anything about whether the underlined sentence (C) holds?
Please pay attention solely to the structure of the sentence (C).

□ NO, as the underlined sentence (C) could hold or not hold.
□ YES, Ida can know something about whether the underlined sentence (C) holds.

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□ NO, as the underlined sentence (C) could hold or not hold.
□ YES, Ida can know something about whether the underlined sentence (C) holds.

And if they answered affirmatively:

What can Ida know about whether the underlined sentence (C) holds?
Please pay attention solely to the structure of the sentence (C).

□ The underlined sentence (C) does NOT hold.
□ The underlined sentence (C) holds.

Method (3/3)

We presented more complex formulae in a two-step, colour-coded fashion:

Ida is waiting in front of the machine and considers the following sentences:
(A) If the next playing block is a *ball*, **then** it is *small*.
(B) If the next playing block is a *ball*, **then** it is **not** *small*.

Now Ida considers the following, combined sentence:
(C) It is not the case, that both (A) and (B).

Or spelled-out:
(C) It is **not** the case, that **both if** the next playing block is a *ball*, **then** it is *small* **and if** the next playing block is a *ball*, **then** it is **not** *small*.

Sample task: Abelard's first principle, $\sim((A \rightarrow B) \wedge (A \rightarrow \sim B))$

Results (1/4)

Table: Response frequencies (in %) in both experiments. The formatting marks predictions by classical logic, *Approach 1* and **Approach 2**.

Name	Holds	Doesn't hold	Can't tell
Introductory examples, $n_1 = 26$			
Excluded Middle	***69.23***	3.85	26.92
Contradiction	3.85	***61.54***	34.62
Contingent Conjunction	—	—	***53.85***
Block 1: *Basic principles*, $n_1 = 26$			
Negated Identity	19.23	**65.38**	*15.38*
Conjunction Elimination	***88.46***	11.54	0.00
Contingent Conditional	11.54	26.92	***61.54***
Self-negated Conditional	0.00	***84.61***	15.38
Identity	***92.31***	0.00	7.69
Arbitrary Fallacy	7.69	15.38	***76.92***
Aristotle's Thesis′	***57.69***	23.08	19.23
Aristotle's Thesis	***53.85***	34.62	11.54

Results (2/4)

Name	Holds	Doesn't hold	Can't tell
Introductory examples, $n_2 + n_3 = 46$			
Excluded Middle	***73.91***	4.35	21.74
Contradiction	6.52	***73.91***	19.57
Contingent Conjunction	—	—	***56.52***
Block 1: *Basic principles*, $n_2 + n_3 = 46$			
Negated Identity	31.74	**63.04**	*15.22*
Conjunction Elimination	***86.96***	2.17	10.87
Contingent Conditional	0.00	8.70	***91.30***
Self-negated Conditional	0.00	***80.43***	19.57
Identity	***86.96***	8.70	4.35
Arbitrary Fallacy	2.17	10.87	***86.96***
Aristotle's Thesis′	***56.52***	30.43	13.04
Aristotle's Thesis	***67.39***	26.09	6.52

Results (3/4)

Name	Holds	Doesn't hold	Can't tell
Block 2: *Conjunctive principles*, $n_1 = 26$			
Negated Abelard's 1st principle	15.38	**65.38**	<u>*19.23*</u>
Contingent Conditionals	15.38	19.23	<u>***65.38***</u>
Abelard's 1st principle	***50.00***	26.92	<u>23.08</u>
Aristotle's second Thesis	30.77	11.54	<u>***57.69***</u>
Contradicting Conditionals	23.08	<u>***46.15***</u>	30.77

Results (4/4)

Name	Holds	Doesn't hold	Can't tell
Block 3: *Iterated principles I*, $n_2 = 21$			
Iterated Self-negated Conditional	4.76	***66.67***	28.57
Boethius' Thesis	***57.14***	28.67	14.29
Iterated Aristotle's Thesis	***47.62***	23.81	28.57
Iterated Identity	***61.90***	4.76	33.33
Reversed Boethius' Thesis	**71.43**	9.52	*19.05*
Boethius Variation 3	28.57	14.29	***57.14***
Improper Transposition (1/2)	14.29	9.52	***76.19***
Block 4: *Iterated principles 2*, $n_3 = 25$			
Iterated Aristotle's Thesis′	***52.00***	12.00	36.00
Improper Transposition (2/2)	8.00	24.00	***68.00***
Denying a Conjunct	0.00	16.00	***84.00***
Boethius' Thesis′	***48.00***	24.00	28.00
Reversed Boethius' Thesis′	**64.00**	16.00	*20.00*
Symmetry	44.00	16.00	***40.00***
Boethius Variation 4	32.00	16.00	***52.00***

- Tension between human intuition and the predictions of classical logic for self-negating conditionals and connexive principles.
- Coherence-based probability semantics validates connexive principles which matches human intuition.
- We tested a variety of relevant propositional formulae in two experiments where participants judged that they hold, that they do not hold or that one cannot tell whether they hold.
- Among the three contestants, Approach 2 had the best agreement with the data, Approach 1 came second and classical logic last by far.
- Sole problematic data: *Symmetry*, $(A \to B) \to (B \to A)$

Other results

Table: Self-assesment (confidence, difficulty and clarity from 0 to 100) and sum of dwell times in experiment 1 ($n = 26$).

Value	Minimum	Mean	SD	Maximum
Confidence	11.00	52.46	23.76	88.00
Difficulty	9.00	42.58	20.26	91.00
Clarity	7.00	59.81	26.95	100.00
Time (in mm:ss)	08:55	14:52	02:54	20:14

Table: Self-assessment (confidence, difficulty and clarity from 0 to 100) and sum of dwell times in experiment 2 ($n_2 + n_3 = 46$).

Value	Minimum	Mean	SD	Maximum
Confidence	5.00	55.24	25.43	91.00
Difficulty	5.00	39.35	16.26	85.00
Clarity	0.00	66.91	26.44	100.00
Time (in mm:ss)	10:06	14:27	02:15	18:22

- Angelo Gilio & Giuseppe Sanfilippo (2014) Conditional random quantities and compounds of conditionals
- Niki Pfeifer & Giuseppe Sanfilippo (2021) Interpreting connexive principles in coherence-based probability logic
- Storrs McCall (2012) A history of connexivity
- Niki Pfeifer (2012) Experiments on Aristotle's Thesis
- Niki Pfeifer & Richard Stöckle-Schobel (2015) Uncertain Conditionals and Counterfactuals in (Non-) Causal Settings
- Niki Pfeifer & Leena Tulkki (2017) Conditionals, counterfactuals, and rational reasoning: An experimental study on basic principles
- Niki Pfeifer & Hiroshi Yama (2017) Counterfactuals, indicative conditionals, and negation under uncertainty: Are there cross-cultural differences?