Belief functions over Belnap–Dunn logic

Sabine FRITTELLA

INSA Centre Val de Loire, Bourges
Laboratoire d'Informatique Fondamentale d'Orléans (LIFO)
sabine.frittella@insa-cvl.fr

Joint work with
Marta Bílková, D. Kozhemiachenko, Ondrej Majer and Sajad Nazari

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1. Representing incomplete/contradictory probabilistic information
   - Belnap–Dunn Logic
   - Non-standard probabilities
2. Dempster-Shafer theory
   - Mass functions, belief functions and plausibility functions
   - Representation of evidence
3. Dempster-Shafer theory and BD logic
   - Our proposal
   - Examples for mass functions, belief functions and combination of evidence
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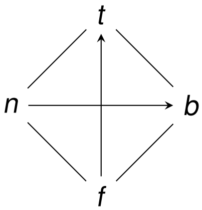
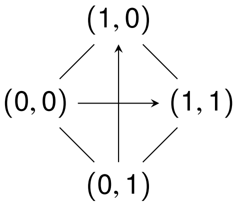
Belnap–Dunn square **4** [Belnap 19]

Belnap–Dunn square $(\mathbf{4}, \wedge, \vee, \neg)$ is a de Morgan algebra.

- $(\mathbf{4}, \wedge, \vee)$ is a lattice
- each element represents the available positive and/or negative information
  - n: no information
  - f: false (is bottom)
  - t: true (is top)
  - b: contradictory information
- $\neg$ is an involutive de Morgan negation.



Belnap–Dunn square 4



Belnap–Dunn Logic: models [Dunn 76]

Language. $L_{\mathrm{BD}} \ni \varphi := p \in \mathrm{Prop} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi$

Models. $M = \langle W,\ v^+, v^- \ :\ W \times \mathrm{Prop} \to \mathbf{2} \rangle$
$v^+(p)$: states where one has information supporting the truth of p
$v^-(p)$: states where one has information supporting the falsity of p

Semantics. Two satisfaction relations $\Vdash^+, \Vdash^-$

$w \vDash^+ p$ iff $w \in v^+(p)$	$w \vDash^- p$ iff $w \in v^-(p)$
$w \vDash^+ \neg\phi$ iff $w \vDash^- \phi$	$w \vDash^- \neg\phi$ iff $w \vDash^+ \phi$
$w \vDash^+ \phi \wedge \phi'$ iff $w \vDash^+ \phi$ and $w \vDash^+ \phi'$	$w \vDash^- \phi \wedge \phi'$ iff $w \vDash^- \phi$ or $w \vDash^- \phi'$
$w \vDash^+ \phi \vee \phi'$ iff $w \vDash^+ \phi$ or $w \vDash^+ \phi'$	$w \vDash^- \phi \vee \phi'$ iff $w \vDash^- \phi$ and $w \vDash^- \phi'$

Non-standard probabilities: frame semantics [Klein et al]

- Independence of the probability assigned to positive and negative information
- Extend BD model with a classical probability measure.

A **probabilistic BD model** is a tuple $M = \langle W, v^+, v^-, m \rangle$, s.t.

- $\langle W, v^+, v^- \rangle$ is a BD model and
- $m : W \to [0, 1]$ is a mass function on W i.e. $\sum_{s \in W} m(s) = 1$

Positive probability of φ: $p^+(\varphi) := \sum \{ m(s) \mid s \Vdash^+ \varphi \}$.
Negative probability of φ: $p^-(\varphi) := \sum \{ m(s) \mid s \Vdash^- \varphi \}$.

Remark. $p^+(\varphi)$ and $p^-(\varphi)$ are independent.

Non-standard probabilities: axioms

[Klein et al] Lemma 1

Let $M = \langle W, v, m \rangle$ be a probabilistic BD frame. Then the non-standard probability function p^+ induced by m satisfies:

(A1) normalization	$0 \leq p^+(\varphi) \leq 1$
(A2) monotonicity	if $\varphi \vdash_{BD} \psi$ then $p^+(\varphi) \leq p^+(\psi)$
(A3) import-export	$p^+(\varphi \wedge \psi) + p^+(\varphi \vee \psi) = p^+(\varphi) + p^+(\psi)$.

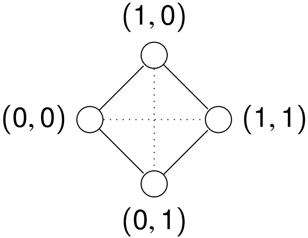
Remarks

- $p^-(\varphi) = p^+(\neg\varphi)$
- Weaker than classical Kolmogorovian axioms.
  Additivity does not hold and is replaced by A3.
- In general $p^+(\neg\varphi) \neq 1 - p^+(\varphi)$
- one can have $0 < p^+(\varphi \wedge \neg\varphi)$

Non-standard probabilities: intuitive representation

Continuous extension of Belnap–Dunn square, which we can see as the product bilattice $\mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$ with $\mathbf{L}_{[0,1]} = ([0,1], \min, \max)$.

- $(p^+(\varphi), p^-(\varphi))$: positive and negative probabilistic support of φ.
- $(0,0)$: no information concerning φ is available
- $(1,1)$: maximally conflicting information
- vertical dashed line: "classical" case



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Belief functions and plausiblity functions

Let $f : \mathcal{P}(S) \to [0,1]$ be a monotone function such that $f(\emptyset) = 0$ and $f(S) = 1$.

- f is a belief function if, for every $k \geq 1$, and for every $A_1, \ldots, A_k \in \mathcal{P}(S)$, it holds that

$$f\left(\bigvee_{1\leq i\leq k} A_i\right) \geq \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot f\left(\bigwedge_{j\in J} A_j\right). \tag{1}$$

- f is a plausibility function if, for every $k \geq 1$, and for every $A_1, \ldots, A_k \in \mathcal{P}(S)$, it holds that

$$f\left(\bigwedge_{1\leq i\leq k} A_i\right) \leq \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot f\left(\bigvee_{j\in J} A_j\right). \tag{2}$$

Mass function

Let $\mathrm{bel}, \mathrm{pl} : \mathcal{P}(S) \to [0,1]$ be a monotone function such that $f(\emptyset) = 0$ and $f(S) = 1$, and $\mathrm{m} : \mathcal{P}(S) \to [0,1]$.

Definition

m is a mass function if $\sum_{A \in \mathcal{P}(S)} \mathrm{m}(A) = 1$.

Theorem

- bel is a belief function iff there is a mass function $\mathrm{m}_{\mathrm{bel}} : \mathcal{P}(S) \to [0,1]$ such that, for every $A \in \mathcal{P}(S)$,

$$\mathrm{bel}(A) = \sum_{B \leq A} \mathrm{m}_{\mathrm{bel}}(B)$$

- if bel is a belief function, then $\mathrm{pl}(A) = 1 - \mathrm{bel}(\neg A)$ is a plausibility function.
- if pl is a plausibility function, then $\mathrm{bel}(A) = 1 - \mathrm{pl}(\neg A)$ is a belief function.

Representation of evidence. Example

- $\mathrm{m} : \mathcal{P}(S) \to [0, 1]$ is computed based on the evidence
- $\mathrm{bel}(A) = \sum_{B \le A} \mathrm{m}(B)$: the evidence supporting a
- $\mathrm{pl}(A) = 1 - \mathrm{bel}(\neg A) = \sum_{B \cap A \neq \emptyset} \mathrm{m}(B)$: the evidence not contradicting A
- $\mathrm{bel}(A) \le \mathrm{pl}(A)$.

Example

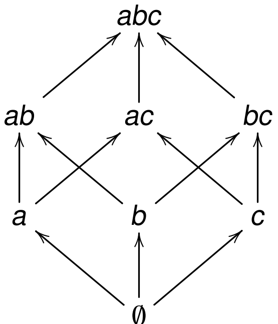
Scenario. A patient has disease a, b or c.
A doctor says "the patient has disease a or b with certainty 0.7."
The doctor gives no information about disease c.

Representation of evidence. An example

Scenario. A patient has disease *a*, *b* or *c*.
A doctor says "the patient has disease *a* or *b* with certainty 0.7."
It is assumed it is impossible for the patient to have two of these diseases.

Representation

- $S = \{a, b, c\}$ and $\mathrm{m}, \mathrm{bel}, \mathrm{pl} : \mathcal{P}(S) \to [0, 1]$
- $\mathrm{m}(\{a, b\}) = 0.7$ and $\mathrm{m}(S) = 0.3$.



An example

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Representation

- $S = \{a, b, c\}$ and $\mathrm{m}, \mathrm{bel}, \mathrm{pl} : \mathcal{P}(S) \rightarrow [0, 1]$
- $\mathrm{m}(\{a, b\}) = 0.7$ and $\mathrm{m}(S) = 0.3$.

We get:

$\mathrm{bel}(\{a\}) = \mathrm{bel}(\{b\}) = \mathrm{bel}(\{c\}) = 0$

$\mathrm{bel}(\{a, b\}) = \sum_{X \subseteq \{a,b\}} \mathrm{m}(X) = 0.7$ $\quad$ $\mathrm{pl}(\{a, b\}) = 1 - \mathrm{bel}(\{c\}) = 1$

$\mathrm{pl}(\{a\}) = \mathrm{pl}(\{b\}) = 1$ $\quad$ $\mathrm{pl}(\{c\}) = 1 - \mathrm{bel}(\{a, b\}) = 0.3$

- $\mathrm{m}(\{a, b\})$: the 'probability' that the disease is in the set $\{a, b\}$ without being able to say to which subset it belongs.
- if m is non-zero only on singletons, then bel and pl are probability functions.

Dempster-Shafer combination rule

Let m_1 and m_2 be two mass functions on a powerset algebra $\mathcal{P}(S)$. Dempster-Shafer combination rule computes their aggregation $\mathrm{m}_{1\oplus 2}$ as follows.

$$\begin{aligned}\mathrm{m}_{1\oplus 2} \;:\; & \mathcal{P}(S) \to [0,1] \\ X \mapsto & \begin{cases} 0 & \text{if } X = \varnothing \\ \dfrac{\sum\{\mathrm{m}_1(X_1)\cdot \mathrm{m}_2(X_2) \mid X_1 \cap X_2 = X\}}{\sum\{\mathrm{m}_1(X_1)\cdot \mathrm{m}_2(X_2) \mid X_1 \cap X_2 \neq \varnothing\}} & \text{otherwise.}\end{cases}\end{aligned}$$

Normalization factor:

$$\begin{aligned}&\sum\{\mathrm{m}_1(X_1)\cdot \mathrm{m}_2(X_2) \mid X_1 \cap X_2 \neq \varnothing\} \\ &= 1 - \sum\{\mathrm{m}_1(X_1)\cdot \mathrm{m}_2(X_2) \mid X_1 \cap X_2 = \varnothing\}\end{aligned}$$

What happens with contradictory evidence?

Scenario

A patient has disease a, b or c.
Doctor 1: "the patient has disease a with certainty 0.9 and disease b with certainty 0.1."
Doctor 2: "the patient has disease c with certainty 0.9 and disease b with certainty 0.1."

Representation

$S = \{a, b, c\}$
$\mathrm{m}_1(\{a\}) = 0.9$ and $\mathrm{m}_1(\{b\}) = 0.1$.
$\mathrm{m}_2(\{c\}) = 0.9$ and $\mathrm{m}_2(\{b\}) = 0.1$.

Dempster-Shafer combination rule gives

$$\mathrm{m}_{1\oplus 2}(\{b\}) = 1$$

because $\{a\} \cap \{b\} = \{a\} \cap \{c\} = \emptyset$.

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Our proposal over BD

Define belief and plausibility on BD-models

Let Prop be a finite set of variables.
$M = \langle W, v^+, v^-, \mathrm{bel}\rangle$ with $\mathrm{bel} : \mathcal{P}(W) \to [0,1]$.

$$\mathrm{bel}^+(\phi) := \mathrm{bel}(|\phi|^+) \quad \text{and} \quad \mathrm{bel}^-(\phi) := \mathrm{bel}(|\phi|^-)$$

bel^+: belief function on the associated Lindenbaum algebra $\mathcal{L}_{\mathrm{BD}}$.
bel^-: belief function on $\mathcal{L}_{\mathrm{BD}}^{op}$.

Remark. if $\perp$ and $\top$ are not in the language bel^+ (resp. pl^+) are general belief (resp. plausibility) functions.
Consequence. $0 \leq \sum_{a \in \mathcal{L}_{\mathrm{BD}}} \mathrm{m}_{\mathrm{bel}^+}(a) \leq 1$

Combination of evidence

Let $\mathcal{L}$ be a finite distributive lattice.

Without $\perp$ and $\top$

$$\begin{aligned} \mathrm{m}_{1\oplus 2} \;:\; & \mathcal{L} \to [0,1] \\ & x \mapsto \sum\{\mathrm{m}_1(x_1)\cdot \mathrm{m}_2(x_2) \mid x_1 \wedge x_2 = x\}. \end{aligned}$$

With $\perp$ and $\top$

$$\begin{aligned} \mathrm{m}_{1\oplus 2} \;:\; & \mathcal{L} \to [0,1] \\ & x \mapsto \begin{cases} 0 & \text{if } x = \perp \\ \dfrac{\sum\{\mathrm{m}_1(x_1)\cdot \mathrm{m}_2(x_2) \mid x_1 \wedge x_2 = x\}}{\sum\{\mathrm{m}_1(x_1)\cdot \mathrm{m}_2(x_2) \mid x_1 \wedge x_2 \neq \perp\}} & \text{otherwise.} \end{cases} \end{aligned}$$

Examples. The two doctors

Scenario. A patient has disease a, b or c.
Doctor 1: a with certainty 0.9 and b with certainty 0.1.
Doctor 2: c with certainty 0.9 and b with certainty 0.1.
Representation. $\mathrm{m}_1, \mathrm{m}_2 \ : \ \mathcal{DM}_3 \to [0,1]$

$$\mathrm{m}_1(x) = \begin{cases} 0.9 & \text{if } x = a \\ 0.1 & \text{if } x = b \\ 0 & \text{otherwise.} \end{cases} \qquad \mathrm{m}_2(x) = \begin{cases} 0.9 & \text{if } x = c \\ 0.1 & \text{if } x = b \\ 0 & \text{otherwise.} \end{cases}$$

Dempster-Shafer combination rule gives

$$\mathrm{m}_{1\oplus 2}(x) = \begin{cases} 0.81 & \text{if } x = a \wedge c \\ 0.09 & \text{if } x = a \wedge b \text{ or } x = b \wedge c \\ 0.01 & \text{if } x = b \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{bel}_{1\oplus 2}(a) = \mathrm{bel}_{1\oplus 2}(c) = 0.9$ and $\mathrm{bel}_{1\oplus 2}(b) = 0.19$

Examples. The two doctors

Representation. $\mathrm{m}_1, \mathrm{m}_2 \; : \; \mathcal{DM}_3 \to [0, 1]$

$$\mathrm{m}_1(x) = \begin{cases} 0.9 & \text{if } x = a \wedge \neg b \\ 0.1 & \text{if } x = \neg a \wedge b \\ 0 & \text{otherwise.} \end{cases} \qquad \mathrm{m}_2(x) = \begin{cases} 0.9 & \text{if } x = \neg b \wedge c \\ 0.1 & \text{if } x = b \wedge \neg c \\ 0 & \text{otherwise.} \end{cases}$$

Dempster-Shafer combination rule gives

$$\mathrm{m}_{1\oplus 2}(x) = \begin{cases} 0.81 & \text{if } x = a \wedge \neg b \wedge c \\ 0.09 & \text{if } x = a \wedge b \wedge \neg b \wedge \neg c \text{ or } x = \neg a \wedge b \wedge \neg b \wedge c \\ 0.01 & \text{if } x = \neg a \wedge b \wedge \neg c \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{bel}_{1\oplus 2}(a) = \mathrm{bel}_{1\oplus 2}(c) = 0.9$ and $\mathrm{bel}_{1\oplus 2}(b) = 0.19$

Examples. The two doctors

Representation. $\mathrm{m}_1, \mathrm{m}_2 \ : \ \mathcal{DM}_3 \to [0,1]$

$$\mathrm{m}_1(x) = \begin{cases} 0.9 & \text{if } x = a \wedge \neg b \wedge \neg c \\ 0.1 & \text{if } x = \neg a \wedge b \wedge \neg c \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{m}_2(x) = \begin{cases} 0.9 & \text{if } x = \neg a \wedge \neg b \wedge c \\ 0.1 & \text{if } x = \neg a \wedge b \wedge \neg c \\ 0 & \text{otherwise.} \end{cases}$$

Dempster-Shafer combination rule gives

$$\mathrm{m}_{1\oplus 2}(x) = \begin{cases} 0.81 & \text{if } x = a \wedge \neg a \wedge \neg b \wedge c \wedge \neg c \\ 0.09 & \text{if } x = a \wedge \neg a \wedge b \wedge \neg b \wedge \neg c \\ & \text{or } x = \neg a \wedge b \wedge \neg b \wedge c \wedge \neg c \\ 0.01 & \text{if } x = \neg a \wedge b \wedge \neg c \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{bel}_{1\oplus 2}(a) = \mathrm{bel}_{1\oplus 2}(c) = 0.9$ and $\mathrm{bel}_{1\oplus 2}(b) = 0.19$

What about plausibility?

- $1 - \text{bel}^+(\neg\phi)$ defines a plausibility function
- $1 - \text{bel}^+(\neg\phi)$ is the sum of the masses of the set of states that do not support the negation of ϕ.
  Problem: a set of states can support both ϕ and $\neg\phi$ and a set of states can support neither.
- In general, we can have $1 - \text{bel}^+(\neg\phi) \leq \text{bel}^+(\phi)$
- We can define plausibility independently of belief and impose $\text{bel}^+(\phi) \leq \text{pl}^+(\phi)$

$M = \langle W, v^+, v^-, \text{bel}, \text{pl} \rangle$ with $\text{bel}, \text{pl} : \mathcal{P}(W) \to [0, 1]$.

$$\begin{array}{rcl} \text{bel}^+(\phi) := \text{bel}(|\phi|^+) & \text{and} & \text{bel}^-(\phi) := \text{bel}(|\phi|^-) \\ \text{pl}^+(\phi) := \text{pl}(|\phi|^+) & \text{and} & \text{pl}^-(\phi) := \text{pl}(|\phi|^-) \end{array}$$

- What kind of two-dimensional reading of belief/plausibility can we propose?
- How can we interpret it?

Two-dimensional interpretation (1/2)

Non-standard probabilities

Models: $(W, v^+, v^-, \mathrm{m} : W \to [0,1])$
$p^+(\phi) = \sum_{s \in |\phi|^+} \mathrm{m}(s)$ and $p^-(\phi) = \sum_{s \in |\phi|^-} \mathrm{m}(s)$

Immediate generalisation for belief.

Non-standard probabilities

Models: $(W, v^+, v^-, \mathrm{bel} : \mathcal{P}(W) \to [0,1])$
$\mathrm{bel}^+(\phi) = \sum_{X \subseteq |\phi|^+} \mathrm{m}(X)$ and $\mathrm{bel}^-(\phi) = \sum_{X \subseteq |\phi|^-} \mathrm{m}(X)$

- $p^+(\phi)$, $\mathrm{bel}^+(\phi)$: the probability/belief that ϕ is true
- $p^-(\phi)$, $\mathrm{bel}^-(\phi)$: the probability/belief that ϕ is false
- bel^+ is monotone and in $[0,1]$, it satisfies the axioms of belief functions instead of
  import-export: $p^+(\varphi \wedge \psi) + p^+(\varphi \vee \psi) = p^+(\varphi) + p^+(\psi)$.

Two-dimensional interpretation (2/2)

Models: $(W, v^+, v^-, \text{bel}, \text{pl} : \mathcal{P}(W) \to [0, 1])$

- Notice that in BD logic $|\phi| = (1, 1)$ reads as: there is evidence that ϕ is true and evidence that ϕ is false.
- In the classical case, $\text{bel}(\phi) = 1 - \text{pl}(\neg\phi)$.
  $\to$ interpretation: $\text{pl}(\neg\phi)$ is the degree of evidence against $\text{bel}(\phi) \to (\text{bel}^+(\phi), \text{pl}^-(\phi))$
  $\to$ $\text{pl}^-(\phi)$ maximum evidence against ϕ we can consider
- Consider both belief $(\text{bel}^+(\phi), \text{bel}^-(\phi))$ and plausibility $(\text{pl}^+(\phi), \text{pl}^-(\phi))$ independently
- If we ask $\text{bel}(X) \leq \text{pl}(X)$, for $X \in \mathcal{P}(W)$, then bel and pl come from different mass functions.
  $\to$ one piece of evidence does not support belief and plausibility in the same manner.
  $\to$ the same piece of evidence gives rise to two mass functions
  e.g., circumstantial evidence vs. direct evidence

Example.

$$s_0 \;:\; \qquad s_1 \;:\; p \qquad s_2 \;:\; \neg p \qquad s_3 \;:\; p, \neg p$$

Assume that $\mathrm{bel}(X) \leq \mathrm{pl}(X)$ and $\mathrm{bel}(|p \wedge \neg p|^{+}) = 1$.

Therfore, we have

$$\sum_{X \subseteq |p \wedge \neg p|^{+}} \mathrm{m}_{\mathrm{bel}}(X) = \sum_{X \subseteq \{s_3\}} \mathrm{m}_{\mathrm{bel}}(X) = \mathrm{m}_{\mathrm{bel}}(\varnothing) + \mathrm{m}_{\mathrm{bel}}(\{s_3\}) = 1.$$

$\mathrm{m}_{\mathrm{bel}}(\varnothing) = 0$, hence $\mathrm{m}_{\mathrm{bel}}(\{s_3\}) = 1$.
We get

$$\begin{aligned} 1 \leq \mathrm{pl}(|p \wedge \neg p|^{+}) &= \sum_{X \not\subseteq |p \wedge \neg p|^{-}} \mathrm{m}_{\mathrm{pl}}(X) = \sum_{X \not\subseteq |p|^{-} \cup |p|^{+}} \mathrm{m}_{\mathrm{pl}}(X) \\ &= \sum_{X \not\subseteq \{s_1, s_2, s_3\}} \mathrm{m}_{\mathrm{pl}}(X) = \mathrm{m}_{\mathrm{pl}}(S) \end{aligned}$$

Therefore, evidence that is strongly persuasive considering $p \wedge \neg p$ is inconclusive regarding the plausibility of either p or $\neg p$.

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