Weak Solutions and Diffuse Interface Models for Incompressible Two-Phase Flows

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December 23, 2015

Abstract

In two-phase flows typically a change of topology arises. This happens when two drops merge or when one bubble splits into two. In such a case the concept of classical solutions to two-phase flow problems, which describe the interface as a smooth hypersurface, breaks down.

This contribution discusses two possible approaches to deal with this problem. First of all weak formulations are discussed which allow for topology changes during the evolution. Such weak formulations involve either varifold solutions, so called renormalized solutions or viscosity solutions.

A second approach replaces the sharp interface by a diffuse interfacial layer which leads to a phase field type representation of the interface. This approach leads typically to quite smooth solutions even when the topology changes.

This contribution introduces the solution concepts, discusses modelling aspects, gives an account of the analytical results known and states how one can recover the sharp interface problem as an asymptotic limit of the diffuse interface problem.

1 Introduction

In the flow of immiscible fluids with interfaces in general topological transitions like droplet break-up and coalescence occur. In such situations classical formulations based on an explicit parameterization break down as singularities will appear at points where the topology changes. This contribution discusses two approaches to deal with this issue. First of all weak formulations of the two-phase flow problem for incompressible fluids are introduced which all allow for singularities in the geometry. The known results for the

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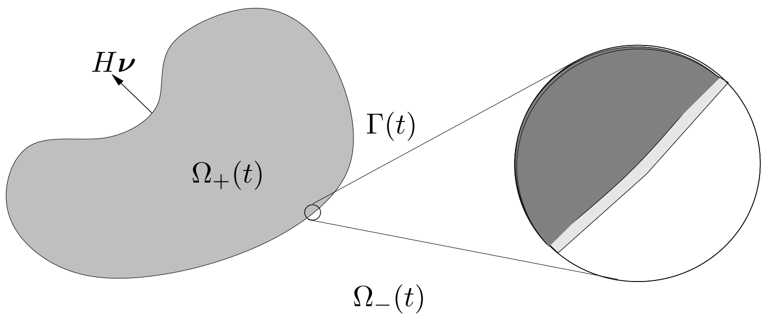

Figure 1: Sharp versus diffuse interface models

different approaches are stated and the advantages and disadvantages of the different formulations are discussed. Secondly diffuse interface methods provide an alternative way to allow for topological transitions. In these models quantities which in traditional sharp interface models are localized to the interfacial surface are now distributed over a diffuse interfacial region. For example, quantities like the density and the viscosity are suitably averaged in the diffuse interface and the surface tension, which is supported on the interface in a sharp interface model, is now a distributed stress within the diffuse interfacial layer, cf. Figure 1.

In the classical sharp interface approach for incompressible, viscous flows the Navier–Stokes equations have to hold in the two phases, described by disjoint open sets $\Omega_\pm(t)$, which are separated by a hypersurface $\Gamma(t)$, which evolves in time. In this contribution we do not allow for slip at the interface which leads to the fact that the tangential part of the fluid velocity does not jump at the interface and we also assume that no phase transitions occur which implies that also the normal part of the velocity does not jump at the interface and that the interface is transported with the fluid velocity. We hence obtain

$$\begin{aligned} [\mathbf{v}]_-^+ &= \mathbf{0}\,, \\ \mathcal{V} &= \mathbf{v}\cdot\boldsymbol{\nu}, \end{aligned}$$

where $\mathbf{v}$ is the fluid velocity, $[.]_-^+$ denotes the jump across the interface $\Gamma(t)$, $\boldsymbol{\nu}$ is a unit normal at the interface $\Gamma(t)$, chosen as interior normal with respect to $\Omega_+(t)$, and $\mathcal{V}$ is the normal velocity.

In addition a tangential stress balance has to hold at the interface. In cases where the interface itself does not produce stresses at the interface the normal stresses have to balance, i.e., on the interface it has to hold

$$\mathbf{T}^+\boldsymbol{\nu} - \mathbf{T}^-\boldsymbol{\nu} = 0 \quad \Leftrightarrow \quad [\mathbf{T}]_-^+\boldsymbol{\nu} = 0$$

where $\mathbf{T}^+$ and $\mathbf{T}^-$ are the values of the stress tensor on both sides of the interface. For a viscous, incompressible fluid the simplest choice for $\mathbf{T}$ is

$\mathbf{T} = 2\eta D\mathbf{v} - p\,\mathrm{Id}$, where $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$. In the following mainly the case with surface tension will be discussed, i.e., surface energy effects are taken into account and in this case the stress balance at the interface is given as the Young–Laplace law

$$[\mathbf{T}]_-^+\boldsymbol{\nu} + \sigma H\boldsymbol{\nu} = 0$$

where σ is the surface tension and H is the mean curvature, which is chosen to be the sum of the principal curvatures with respect to $\boldsymbol{\nu}$.

For weak formulations it is often convenient to reformulate the fact that the interface is transported with the fluid velocity. Defining χ as the characteristic function of one of the phases we can formally rewrite the equation $\mathcal{V} = \mathbf{v}\cdot\boldsymbol{\nu}$ as

$$\partial_t\chi + \mathbf{v}\cdot\nabla\chi = 0$$

which for a velocity field that has zero divergence is formally equivalent to

$$\partial_t\chi + \mathrm{div}(\mathbf{v}\chi) = 0\,.$$

Of course the last two equations need to be interpreted in a suitable weak sense and different formulations will be discussed in Section 2. All these formulations will allow for singularities of the interface and in particular allow for topological transitions. The weak formulations mentioned above are

- approaches based on the theory of viscosity solutions, see [46, 76, 78],
- methods which use the concept of renormalized solutions of transport equations, see [62, 63].

These approaches work for the case without surface tension. If surface tension effects are present also the mean curvature has to be interpreted in a weak sense and this can be done in the context of varifolds, see [2, 14, 16, 22, 64, 79] and Section 2.2.

In diffuse interface models (which are also called phase field models) the sharp interface is replaced by an interfacial layer of finite width and a smooth order parameter is used to distinguish between the two bulk fluids and the diffuse interface. The order parameter takes distinct constant values in each of the bulk fluids and varies smoothly across the thin interfacial layer. In the sharp interface case with surface tension the total energy is given as

$$\int_\Omega \frac{\rho}{2}|\mathbf{v}|^2 dx + \sigma\mathcal{H}^{d-1}(\Gamma)$$

where Ω is the domain occupied by the fluid, ρ is the mass density, Γ is the interface and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional surface measure. The first term is the kinetic energy and the second term accounts for interfacial energy. It is well-known based on work of van der Waals, Korteweg, Cahn and Hilliard that interfacial energy and also related capillary forces can be modeled with the help of density variables which vary continuously across the interface. In these approaches the term $\sigma\mathcal{H}^{d-1}(\Gamma)$ is replaced by a multiple of

$$\mathcal{F}(\varphi) := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\right) dx \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, φ is an order parameter taking the values ± 1 in the two phases and ψ is a double well potential which simplest form is $\psi(\varphi) = \frac{1}{4}(1-\varphi^2)^2$. One can now try to model the physics at the interface with the help of φ and would obtain a new problem which should approximate the above sharp interface problem.

As new energy one obtains

$$\int_\Omega \frac{\rho(\varphi)}{2}|\mathbf{v}|^2 dx + \hat{\sigma}\int_\Omega \left(\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\right) dx\,, \quad \hat{\sigma} > 0\,,$$

and the transport equation becomes

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = m_\varepsilon\Delta\mu$$

with

$$\mu = \hat{\sigma}\frac{\delta\mathcal{F}}{\delta\varphi}$$

where $\frac{\delta\mathcal{F}}{\delta\varphi}$ is the first variation of $\mathcal{F}$.

One obtains the law $\mathcal{V} = \mathbf{v}\cdot\boldsymbol{\nu}$ from the free boundary problem in the limit $\varepsilon \to 0$ by choosing $m_\varepsilon \sim \varepsilon$, see Section 4. It will turn out, see Section 3, that the term

$$\sigma H\boldsymbol{\nu}\, d\mathcal{H}^{d-1}$$

contributing to the stress balance at the interface will become a multiple of

$$\mu\nabla\varphi$$

which is a term which is distributed over the diffuse interfacial layer. In the simplest case the momentum balance in the Navier–Stokes equation can be written as

$$\rho\partial_t\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \operatorname{div}\mathbf{T} = \hat{\sigma}\varepsilon\nabla\cdot\left(|\nabla\varphi|^2\left(\mathrm{Id} - \frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)\right)$$

where the term $\mathrm{Id} - \frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}$ corresponds to $\mathrm{Id} - \boldsymbol{\nu}\otimes\boldsymbol{\nu}$ which is a multiple of the classical interfacial stress tensor which is just the projection onto the tangent space, cf. weak formulation (2.16) below.

2 Weak Formulations

In this section we discuss different notions of weak/generalized solutions of the two-phase flow of two incompressible, immiscible Newtonian fluids inside a bounded domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$. The fluids fill disjoint domains $\Omega_+(t)$ and $\Omega_-(t)$, $t > 0$, and the interface between both fluids is denoted by $\Gamma(t) = \partial\Omega_\pm(t)$. We assume that $\Gamma(t)$ is compactly contained in Ω, which means that we exclude flows, where a contact angle problem occurs. Hence $\Omega = \Omega_+(t) \cup \Omega_-(t) \cup \Gamma(t)$. The flow is described using the velocity $\mathbf{v}\colon \Omega \times (0,\infty) \to \mathbb{R}^d$ and the pressure $p\colon \Omega \times (0,\infty) \to \mathbb{R}$ in both fluids in Eulerian coordinates. We consider the cases with and without surface tension at the interface. Precise assumptions are made below. Under suitable smoothness assumptions, the flow is obtained as solution of the system

$$\rho_\pm\partial_t\mathbf{v} + \rho_\pm\mathbf{v}\cdot\nabla\mathbf{v} - \eta_\pm\Delta\mathbf{v} + \nabla p = 0 \qquad \text{in } \Omega_\pm(t), t > 0, \tag{2.1}$$

$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in } \Omega_\pm(t), t > 0, \tag{2.2}$$

$$[\mathbf{v}]_-^+ = \mathbf{0} \qquad \text{on } \Gamma(t), t > 0, \tag{2.3}$$

$$-[2\eta D\mathbf{v}]_-^+\boldsymbol{\nu} + [p]_-^+\boldsymbol{\nu} = \sigma H\boldsymbol{\nu} \qquad \text{on } \Gamma(t), t > 0, \tag{2.4}$$

$$\mathcal{V} = \boldsymbol{\nu}\cdot\mathbf{v} \qquad \text{on } \Gamma(t), t > 0, \tag{2.5}$$

$$\mathbf{v} = \mathbf{0} \qquad \text{on } \partial\Omega, t > 0, \tag{2.6}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0 \qquad \text{in } \Omega, \tag{2.7}$$

together with $\Omega_+(0) = \Omega_0^+$. Here $\mathcal{V}$ and H denote the normal velocity and mean curvature of $\Gamma(t)$, resp., taken with respect to the interior normal $\boldsymbol{\nu}$ of $\partial\Omega_+(t) = \Gamma(t)$, $\sigma \geq 0$ is the surface tension constant ($\sigma = 0$ means that no surface tension is present), $\rho_\pm > 0$ and $\eta_\pm > 0$ are the (constant) densities and viscosities of the fluids, respectively. We note that (2.1)-(2.2) describe the conservation of linear momentum and mass for both fluids. Furthermore, (2.3) is a no-slip boundary condition at $\Gamma(t)$, implying continuity of $\mathbf{v}$ across Γ, (2.4) is the balance of forces at the boundary, (2.5) is the kinematic condition that the interface is transported with the flow of the mass particles, and (2.6) is the no-slip condition at the boundary of Ω. Here exterior forces are neglected for simplicity.

There are many results on well-posedness locally in time or global existence close to equilibrium states for quite regular solutions of this two-phase flow and similar free boundary value problems for viscous incompressible fluids. We refer to Solonnikov [73, 75], Beale [25, 26], Tani and Tanaka [77], Shibata and Shimizu [70] or Shibata and Shimizu [71] and the references given there. These approaches are a priori limited to flows, in which the interface does not develop singularities and the domain filled by the fluid does not change its topology.

In the following we discuss different notions of generalized solutions, which allow for singularities of the interface and which exist globally in time for general initial data. A similar and more detailed discussion can be found in [2]. To this end, we first need a suitable weak formulation of the system above. By multiplication of (2.1) with a divergence free vector field $\boldsymbol{\varphi}$ and integration by parts using in particular the jump relation (2.5), one obtains

$$
\begin{aligned}
&-\int_0^\infty \int_\Omega \rho(\chi)\mathbf{v}\cdot\partial_t\boldsymbol{\varphi}\,dx\,dt - \int_\Omega \rho(\chi_0)\mathbf{v}_0\cdot\boldsymbol{\varphi}|_{t=0}\,dx \\
&+\int_0^\infty \int_\Omega \rho(\chi)(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\boldsymbol{\varphi}\,dx\,dt + \int_0^\infty \int_\Omega 2\eta(\chi)D\mathbf{v} : D\boldsymbol{\varphi}\,dx\,dt \\
&= \sigma\int_0^\infty \left\langle H_{\Gamma(t)}, \boldsymbol{\varphi}(t)\right\rangle\,dt \qquad (2.8)
\end{aligned}
$$

for all $\boldsymbol{\varphi} \in C^\infty_{(0)}(\Omega\times[0,\infty))^d$ with $\operatorname{div}\boldsymbol{\varphi} = 0$, where $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$, $\chi(x,t) = \chi_{\Omega_+(t)}(x)$ for all $x\in\Omega$, $t>0$, $\chi_0 = \chi_{\Omega_0^+}$, χ_A denotes the characteristic function of a set A, $\rho(1) = \rho_+$, $\rho(0) = \rho_-$, $\eta(1) = \eta_+$, $\eta(0) = \eta_-$, and

$$
\left\langle H_{\Gamma(t)}, \boldsymbol{\varphi}(t)\right\rangle := \int_{\Gamma(t)} H(x,t)\boldsymbol{\nu}(x)\cdot\boldsymbol{\varphi}(x,t)\,d\mathcal{H}^{d-1}(x). \qquad (2.9)
$$

Here $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. Now, if $\mathbf{v}$ and Γ are sufficiently smooth, one obtains by choosing $\boldsymbol{\varphi} = \mathbf{v}$ the *energy inequality*

$$
\begin{aligned}
&\int_\Omega \frac{\rho(\chi(x,T))|\mathbf{v}(x,T)|^2}{2}\,dx + \sigma\mathcal{H}^{d-1}(\Gamma(T)) \qquad (2.10)\\
&\quad+\int_0^T\int_\Omega 2\eta(\chi)|D\mathbf{v}|^2\,dx\,dt \le \int_\Omega \frac{\rho(\chi|_{t=0})|\mathbf{v}_0|^2}{2}\,dx + \sigma\mathcal{H}^{d-1}(\Gamma_0)
\end{aligned}
$$

for all $T>0$ (even with equality), where $\Gamma_0 = \partial\Omega_0^+$. Here we have used

$$
\frac{d}{dt}\mathcal{H}^{d-1}(\Gamma(t)) = -\int_{\Gamma(t)} H\mathcal{V}\,d\mathcal{H}^{d-1} = -\langle H_{\Gamma(t)}, \mathbf{v}(t)\rangle \qquad (2.11)
$$

due to (2.5), cf. [47, Equation 10.12]. More details for a more general model can be found in [12, Section 5].

Since $\eta_\pm, \rho_\pm > 0$, (2.10) yields the a priori estimate

$$
\mathbf{v}\in L^\infty(0,\infty;L^2_\sigma(\Omega)) \quad\text{and}\quad D\mathbf{v}\in L^2(\Omega\times(0,\infty))^{d\times d} \qquad (2.12)
$$

for any sufficiently smooth solution of (2.1)-(2.7). Here $L^p(M)$, $1\le p\le\infty$, denotes the usual Lebesgue space, $L^p_{loc}(M)$ its local and $L^p(M;X)$ its vector-valued analog for a given Banach space X. Moreover, if $A\subset\mathbb{R}$, then $L^p(M;A)$ consists of all $f\in L^p(M)$ with $f(x)\in A$ for a.e. $x\in M$. Finally, $L^p_\sigma(\Omega) = \overline{\{\boldsymbol{\varphi}\in C_0^\infty(\Omega)^d : \operatorname{div}\boldsymbol{\varphi} = 0\}}^{L^2(\Omega)}$ is the set of all weakly divergence free vector fields $f\in L^p(\Omega)^d$.

As will be shown below, if $\sigma > 0$, then (2.10) yields an a priori bound of

$$\chi \in L^\infty(0,\infty; BV(\Omega)),$$

where $BV(\Omega) = \{f \in L^1(\Omega) : \nabla f \in \mathcal{M}(\Omega)\}$ denotes the space of functions with bounded variation, cf., e.g., [23, 43] and $\mathcal{M}(\Omega) = C_0(\Omega)'$ is the space of finite Radon measures. In the case without surface tension, i.e., $\sigma = 0$, we only obtain that $\chi \in L^\infty(Q)$ is a priori bounded by one. This motivates to look for weak solutions $(\mathbf{v}, \chi)$ lying in the function spaces above, satisfying (2.10) with a suitable substitute of (2.9), such that $(\mathbf{v}, \chi)$ solve (2.8) as well as the transport equation

$$\partial_t \chi + \mathbf{v} \cdot \nabla \chi = 0 \qquad \text{in } Q, \tag{2.13}$$

$$\chi|_{t=0} = \chi_0 \qquad \text{in } \Omega \tag{2.14}$$

for $\chi_0 = \chi_{\Omega_0^+}$ in a suitable weak sense. Note that (2.13) is a weak formulation of (2.5), cf. [62, Lemma 1.2].

2.1 Two-Phase Flow without Surface Tension

Throughout this subsection we assume that $\sigma = 0$, i.e., no surface tension is present. Then the two-phase flow consists of a coupled system of the Navier–Stokes equation with variable viscosities and a transport equation for the characteristic function $\chi(t) = \chi_{\Omega_+(t)}$. Then this is a special case of the so-called density-dependent Navier–Stokes equation, cf., e.g., Desjardins [39] and references given there. For given χ it is not difficult to construct a weak solution of the Navier–Stokes equation (2.8) with the aid of a suitable approximation scheme (e.g., Galerkin approximation). New difficulties arise due to the mean curvature term $\langle H_{\Gamma(t)}, .\rangle$, which depends non-linearly on the normal of $\Gamma(t)$.

For the coupled system (2.8) together with (2.13)-(2.14) there are two different approaches. The essential difference is in which sense the transport equation is solved. One approach is due to Giga and Takahashi [46], who solved (2.13)-(2.14) in the sense of viscosity solutions, where the characteristic functions $(\chi(t), \chi_0)$ are replaced by continuous level-set functions $(\psi(t), \psi_0)$ such that

$$\Omega_0^\pm = \{x \in \Omega : \psi_0(x) \gtrless 0\}.$$

For simplicity they consider periodic boundary conditions, i.e., $\Omega = \mathbb{T}^d$. Since v is in general not Lipschitz continuous, the existence of a viscosity solution of (2.13)-(2.14) with (χ, χ_0) replaced by continuous level-set functions (ψ, ψ_0) is not known. There are only a least super-solution $\psi^+(t)$ and a largest sub-solution $\psi^-(t)$ of the transport equation. Then one defines

$$\Omega_\pm(t) = \left\{x \in \Omega : \psi^\pm(x,t) \gtrless 0\right\}.$$

With this definition $\Omega_\pm(t)$ are disjoint open sets but the "boundary" $\Gamma(t) = \mathbb{T}^d \setminus (\Omega_+(t) \cup \Omega_-(t))$ might have interior points and might have positive Lebesgue's measure. Giga and Takahashi call this possible effect "boundary fattening". With this definition they construct weak solutions of a two-phase Stokes flow, i.e., the convective term $\mathbf{v} \cdot \nabla \mathbf{v}$ is neglected in (2.8), assuming that the viscosity difference $|\eta_+ - \eta_-|$ is sufficiently small and $\rho_+ = \rho_-$; cf. [46] for details. This approach was adapted to the case of a Navier–Stokes two-phase flow by Takahashi [76] under similar assumptions and to a one-phase flow for an ideal, irrotational and incompressible fluid by Wagner [78].

The other approach was established by Nouri and Poupaud [62] and Nouri et. al. [63] and is based on the results of DiPerna and Lions [42] on renormalized solutions of the transport equation (2.13)-(2.14) for a velocity field $\mathbf{v}$ with bounded divergence. Here $\chi \in L^\infty(Q)$ is called a of (2.13)-(2.14) if for all $\beta \in C^1(\mathbb{R})$ which vanish near 0 the function $\beta(\chi)$ solves (2.13)-(2.14) with initial values $\beta(\chi_0)$, cf. [42] for details. In particular, this implies that $\chi(t,x) \in \overline{\{\chi_0(x) : x \in \Omega\}}$ for almost all $t > 0, x \in \Omega$. Due to [42, Theorem II.3], for every $\chi_0 \in L^\infty(\mathbb{R}^d)$ there is a *unique* renormalized solution of (2.13)-(2.14) under general conditions on $\mathbf{v}$, which are weaker than the condition (2.12). Based on this notion the following result for the two-phase flow without surface tension holds true:

Theorem 2.1. *For every* $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $\chi_0 \in L^\infty(\Omega; \{0,1\})$ *there are* $\mathbf{v} \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; H^1_0(\Omega)^d)$ *and* $\chi \in L^\infty(Q; \{0,1\})$ *that are a weak solution of the two-phase flow* (2.1)-(2.7) *without surface tension* ($\sigma = 0$) *in the sense that* (2.8) *holds true for all* $\boldsymbol{\varphi} \in C^\infty_{(0)}(\Omega \times [0,\infty))^d$ *with* $\operatorname{div} \boldsymbol{\varphi} = 0$, χ *is the unique renormalized solution of the transport equation of (2.13)-(2.14), and (2.10) holds for almost all* $t > 0$ *with* $\sigma = 0$.

The result was proved by Nouri and Poupaud [62] for the case of a bounded domain Ω with Lipschitz boundary. The authors even considered the case of a multi-phase flow with more then two components. The result was extended to generalized Newtonian fluids of power law type for an power-law exponent $q \geq \frac{2d}{d+2} + 1$ in [3].

In order to prove the latter theorem, a key step is to show strong compactness of the sequence χ_k in $L^p(Q_T)$, $1 \leq p < \infty$, where $Q_T = \Omega \times (0,T), T > 0$, and $(\mathbf{v}_k, \chi_k)$ is a suitably constructed approximation sequence. This is done by using the fact that

$$\|\chi_k(t)\|^p_{L^p(\Omega)} = \int_\Omega \chi_k(t,x)\, dx = \int_\Omega \chi_0(x)\, dx$$

if χ_k are solutions of (2.13)-(2.14) with $\mathbf{v}$ replaced by $\mathbf{v}_k$ and $\operatorname{div} \mathbf{v}_k = 0$. Using that

$$\begin{aligned} \chi_k &\rightharpoonup^*_{k\to\infty} \chi && \text{in } L^\infty(Q), \\ \nabla \mathbf{v}_k &\rightharpoonup^*_{k\to\infty} \nabla \mathbf{v} && \text{in } L^2(Q) \end{aligned}$$

for a suitable subsequence one shows that χ solves (2.13)-(2.14), cf. [3, Lemma 5.1]. Here $\rightharpoonup^*$ denotes the weak-$*$ convergence. Therefore

$$\|\chi(t)\|^p_{L^p(\Omega)} = \int_\Omega \chi(t,x)\,dx = \int_\Omega \chi_0(x)\,dx = \|\chi_k(t)\|^p_{L^p(\Omega)}.$$

This implies strong convergence $\chi_k \to_{k\to\infty} \chi$ in $L^p(Q_T)$, $1 \le p < \infty$, for every $T > 0$. Based on this, one can pass to the limit in all terms in (2.8).

Remark 2.2. *Using the solution of Theorem 2.1, we can define the sets* $\Omega_+(t) = \{x \in \Omega : \chi(t) = 1\}$ *and* $\Omega_-(t) = \{x \in \Omega : \chi(t) = 0\}$. *Then we know that* $|\Omega_+(t)| = |\Omega_0^+|$ *and* $\Omega\setminus(\Omega_+(t)\cup\Omega_-(t))$ *has Lebesgue measure zero. But, since only* $\chi \in L^\infty(Q)$ *is known, it is not clear whether* $\Omega_\pm(t)$ *have interior points. In particular, it is not excluded that* $\overline{\Omega_+(t)} = \Omega$ *and* $\operatorname{int}\Omega_+(t) = \emptyset$. *Therefore it is not immediately clear what the "interface" between both fluids should be. If one naively defines the interface as* $\Gamma(t) = \partial\Omega_+(t)$, *then* $\Gamma(t)$ *can have positive Lebesgue measure as in the result by Giga and Takahasi.*

It seems that by neglecting surface tension in the two phase flow, one looses a "good control" of the interface between both fluids. At least the precise regularity of the interface seems to be unknown in general. Some results in this direction can be found in the contribution by Danchin and Mucha [37], where existence and uniqueness of more regular solutions for the inhomogeneous Navier–Stokes equation with discontinuous initial density is shown under several smallness assumptions

2.2 Case with Surface Tension: Varifold Solutions

As discussed in the previous section, a deficit of the two-phase flow without surface tension is that there is no good information on the properties of the interface. As mentioned in the introduction, if $\sigma > 0$, the energy equality (2.10) for sufficiently smooth solutions provides an a priori estimate of the interface:

$$\sup_{0\le t<\infty} \mathcal{H}^{d-1}(\Gamma(t)) \le \left(\frac{1}{2\sigma}\|v_0\|_2^2 + \mathcal{H}^{d-1}(\Gamma_0)\right). \tag{2.15}$$

This implies an a priori bound of χ in the space $BV(\Omega)$ as follows: Note that, if $\Gamma(t) = \partial\Omega_+(t)$ is sufficiently smooth, Gauss' theorem yields

$$-\langle\nabla\chi(t),\boldsymbol{\varphi}\rangle = \int_{\Omega_+(t)} \operatorname{div}\boldsymbol{\varphi}(x)\,dx = -\int_{\Gamma(t)} \boldsymbol{\nu}\cdot\boldsymbol{\varphi}(x)\,d\mathcal{H}^{d-1}(x)$$

for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^d$. Hence the distributional gradient $\nabla\chi(t)$ is a finite Radon measure and

$$\|\nabla\chi(t)\|_{\mathcal{M}(\Omega)} = \mathcal{H}^{d-1}(\Gamma(t)).$$

Thus, if $\sigma > 0$, then $\chi(t) \in BV(\Omega)$ for all $t > 0$ and (2.15) gives an a priori estimate of

$$\chi \in L^\infty(0,\infty; BV(\Omega)).$$

Conversely, if $\chi(t) = \chi_E \in BV(\Omega)$ for some set $E = E(t)$, then E is said to be of *finite perimeter* and the following characterisation holds, cf. [43, Section 5.7, Theorem 2]:

$$\langle \nabla\chi(t), \boldsymbol{\varphi}\rangle = \int_{\partial^* E} \boldsymbol{\nu}_E \cdot \boldsymbol{\varphi}(x)\, d\mathcal{H}^{d-1}(x),$$

where $\partial^* E$ is the *reduced boundary of* E, cf. [43, Definition 5.7], $\boldsymbol{\nu}_E = \frac{\nabla\chi_E}{|\nabla\chi_E|}$, and $\partial^* E$ is *countably* $(d-1)$*-rectifiable* in the sense that

$$\partial^* E = \bigcup_{k=1}^\infty K_k \cup N,$$

where K_k are compact subsets of C^1-hypersurfaces S_k, $k \in \mathbb{N}$, $\mathcal{H}^{d-1}(N) = 0$, and $\nu_E|_{S_k}$ is normal to S_k. Moreover, by [43, Section 5.8, Lemma 1] $\partial_* E \subseteq \partial^* E$ and $\mathcal{H}^{d-1}(\partial^* E \setminus \partial_* E)$, where $\partial_* E$ is the *measure theoretic boundary* of E consisting of all $x \in \Omega$ such that

$$\limsup_{r\to 0} \frac{\mathcal{L}^d(B(x,r)\cap E)}{r^d} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mathcal{L}^d(B(x,r)\setminus E)}{r^d} > 0,$$

where $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$.

Based on these properties, one can define the mean curvature functional of a set of finite perimeter E as

$$\langle H_{\partial^* E}, \varphi\rangle \equiv \langle H_{\chi_E}, \varphi\rangle := -\int_{\partial^* E} \operatorname{tr}(P_\tau \nabla\varphi)\, d\mathcal{H}^{d-1}, \qquad \varphi \in C_0^1(\Omega)^d, \quad (2.16)$$

where $P_\tau = I - \nu_E(x) \otimes \nu_E(x)$. Note that $\operatorname{tr}(P_\tau\nabla\varphi)$ corresponds to the divergence of φ along the "surface" $\partial^* E$ and that by integration by parts (2.16) coincides with the usual definition if $\partial^* E$ is a C^2-surface, cf., e.g., Giusti [47, Chapter 10].

In the following we will assume that $\rho_+ = \rho_- = 1$ and $\Omega = \mathbb{R}^d$. Motivated by the considerations above, we define weak solutions of the two-phase flow in the case of surface tension as follows:

Definition 2.3. (Weak Solutions)
Let $\sigma > 0$. Then

$$\mathbf{v} \in L^\infty(0,\infty; L^2_\sigma(\mathbb{R}^d)) \cap L^2(0,\infty; H^1_0(\mathbb{R}^d)^d),$$
$$\chi \in L^\infty_{\omega*}(0,\infty; BV(\mathbb{R}^d;\{0,1\})),$$

are called a weak solution of the two-phase flow *for initial data* $\mathbf{v}_0 \in L^2_\sigma(\mathbb{R}^d)$, $\chi_0 = \chi_{\Omega_0^+}$ *for a bounded domain* $\Omega_0^+ \subset\subset \mathbb{R}^d$ *of finite perimeter if the following conditions are satisfied:*

(*i*) *(2.8) holds for all* $\boldsymbol{\varphi} \in C^\infty_{(0)}(\mathbb{R}^d \times [0,\infty))^d$ *with* $\operatorname{div} \boldsymbol{\varphi} = 0$*, where* $H_{\Gamma(t)}$ *is replaced by* $H_{\chi(t)}$ *defined as in (2.16).*

(*ii*) χ *is a the renormalized solution of* (2.13)-(2.14).

(*iii*) *The energy inequality*

$$\frac{1}{2}\|\mathbf{v}(t)\|_2^2 + \sigma\|\nabla\chi(t)\|_{\mathcal{M}} + \int_{Q_t} 2\eta(\chi)|D\mathbf{v}|^2 \, d(x,\tau) \leq \frac{1}{2}\|v_0\|_2^2 + \sigma\|\nabla\chi_0\|_{\mathcal{M}} \tag{2.17}$$

holds for almost all $t \in (0,\infty)$.

Unfortunately, the existence of weak solutions as defined above is open. The reason are possible oscillation and concentration effects related to the interface, which cannot be excluded so far. This prevents us from passing to the limit in the mean curvature functional (2.16) during an approximation procedure used to construct weak solutions.

In order to demonstrate these effects, let E_k be a sequence of sets of finite perimeter such that $\chi_k \equiv \chi_{E_k}$ is bounded in $BV(\Omega)$ and let $\Omega = \mathbb{R}^d$. Then after passing to a suitable subsequence, we can assume that

$$\begin{aligned} \chi_k &\to_{k\to\infty} \chi && \text{in } L^1_{loc}(\mathbb{R}^d), \\ \nabla\chi_k &\rightharpoonup^*_{k\to\infty} \nabla\chi && \text{in } \mathcal{M}(\mathbb{R}^d), \\ |\nabla\chi_k| &\rightharpoonup^*_{k\to\infty} \mu && \text{in } \mathcal{M}(\mathbb{R}^d). \end{aligned}$$

But then the question arises how $|\nabla\chi|$ and μ are related and whether

$$\lim_{k\to\infty} \langle H_{\chi_{\varepsilon_k}}, \psi\rangle = \langle H_\chi, \psi\rangle \tag{2.18}$$

holds. The continuity result due Reshetnyak, cf. [23, Theorem 2.39], gives a sufficient condition for (2.18): If

$$\lim_{k\to\infty} |\nabla\chi_k|(\mathbb{R}^d) = |\nabla\chi|(\mathbb{R}^d), \tag{2.19}$$

then (2.18) holds. But in general (2.19) will not hold for example because of the following oscillation/concentration effects at the reduced boundary of E:

(*i*) Several parts of the boundary $\partial^* E_k$ might meet.

(*ii*) Oscillations of the boundary might reduce the area in the limit.

(*iii*) There might be an “infinitesimal emulsion”.

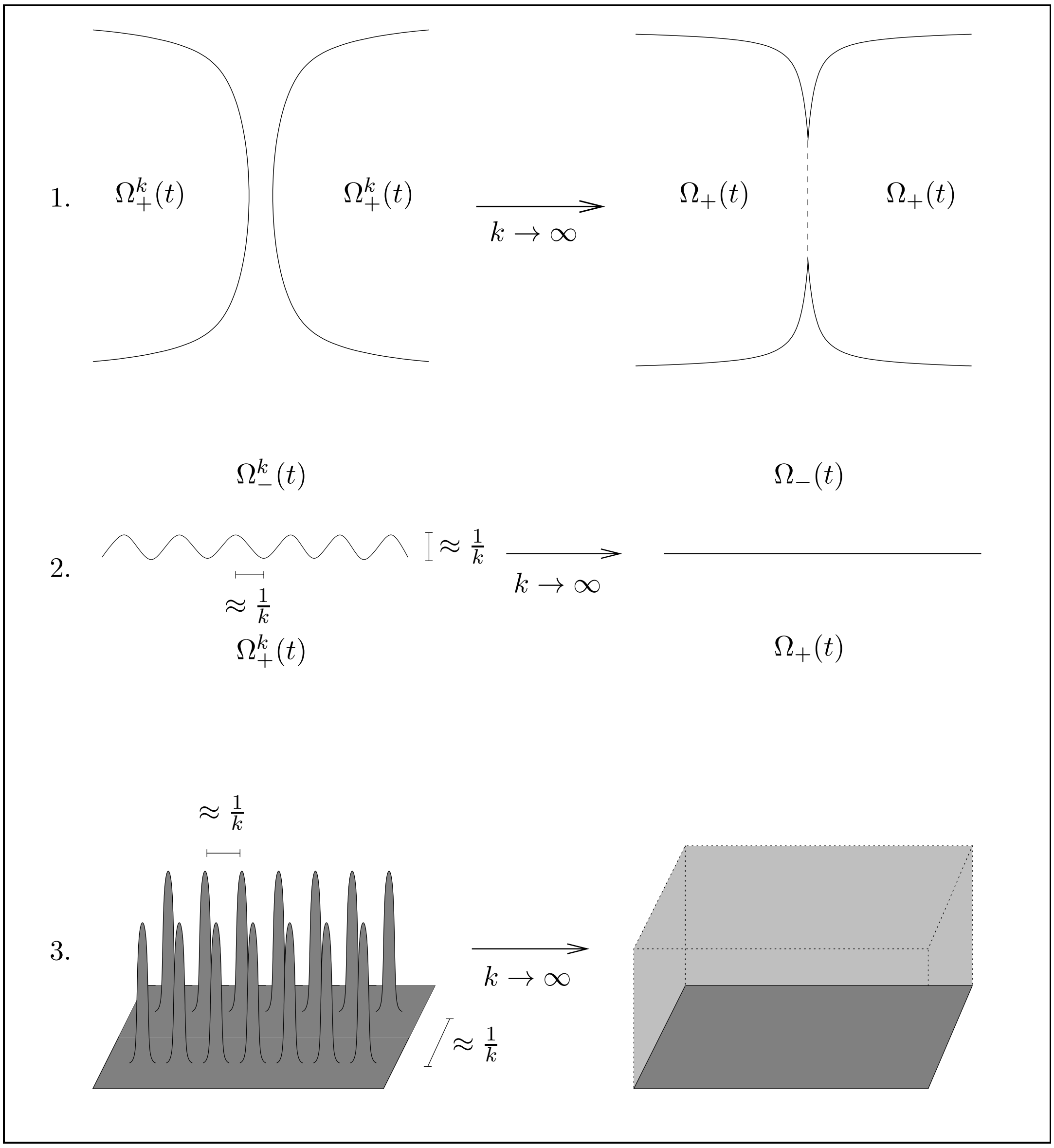

Figure 2: Some possible oscillation/concentration effect

These effects are sketched in Figure 2.

It is an open problem how to exclude such kind of oscillation/concentration effects. – This might even not be possible in general since the model might not describe the behavior of both fluids appropriately when, e.g., a lot of small scale drops are forming. – One way out of this problem is to define so-called *varifold solution* of a two-phase flow, which was first done by Plotnikov [64] in the case of $d = 2$ for shear-thickening non-Newtonian fluids. Here a general (oriented) varifold V on a domain Ω is simply a non-negative measure in $\mathcal{M}(\Omega \times \mathbb{S}^{d-1})$, where $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$. By disintegration, cf. [23, Theorem 2.28], a varifold V can be decomposed in a non-negative measure $|V| \in \mathcal{M}(\Omega)$ and a family of probability measures $V_x \in \mathcal{M}(\mathbb{S}^{d-1})$, $x \in \Omega$, such that

$$\langle V, \psi \rangle = \int_\Omega \int_{\mathbb{S}^{d-1}} \psi(x,s)\, dV_x(s)\, d|V|(x) \qquad \text{for all } \psi \in C_0(\Omega \times \mathbb{S}^{d-1}).$$

Morevoer, $|V|$ corresponds to the measure of the "area of the interface" and V_x defines a probability for the "normal at the interface" for $|V|$-a.e. x.

The reduced boundary $\partial^* E$ of a set of finite perimeter induces naturally a varifold by setting $|V| = |\nabla \chi_E|$ and $V_x = \delta_{\nu_E(x)}$ for $x \in \partial^* E$, where δ_ν denotes the Dirac measure at $\nu \in \mathbb{S}^{d-1}$. Hence the associated varifold V_E is

$$\langle V_E, \psi \rangle = \int_\Omega \psi(x, \nu_E(x))\, d|V|(x) \qquad \text{for all } \psi \in C_0(\Omega \times \mathbb{S}^{d-1}).$$

Now let E_k be a sequence of sets of finite perimeter as above. Then by the weak-$*$ compactness of $\mathcal{M}(\Omega \times \mathbb{S}^{d-1})$, there is a limit varifold $V \in \mathcal{M}(\Omega \times \mathbb{S}^{d-1})$ such that

$$\langle V, \psi \rangle = \lim_{k\to\infty} \langle V_{E_k}, \psi \rangle \qquad \text{for all } \psi \in C_0(\Omega \times \mathbb{S}^{d-1})$$

for a suitable subsequence. Hence using $\psi(s,x) = \operatorname{tr}((I - s \otimes s)\nabla \boldsymbol{\varphi}(x))$ for $\boldsymbol{\varphi} \in C_0^1(\Omega)^d$ we conclude that

$$\lim_{k\to\infty} \langle H_{\chi_{E_k}}, \psi \rangle = \int_{\Omega\times\mathbb{S}^{d-1}} \operatorname{tr}((I - s\otimes s)\nabla \boldsymbol{\varphi}(x))\, dV(s,x) =: -\langle \delta V, \boldsymbol{\varphi} \rangle \quad (2.20)$$

for all $\boldsymbol{\varphi} \in C_0^1(\Omega)^d$. Here $\delta V \in C_0^1(\Omega; \mathbb{R}^d)'$ defined as above is called the *first variation* of the generalized varifold V. Moreover,

$$\begin{aligned} -\langle \nabla \chi_E, \boldsymbol{\varphi} \rangle &= -\lim_{k\to\infty} \langle \nabla \chi_{E_k}, \boldsymbol{\varphi} \rangle \\ &= \lim_{k\to\infty} \int_\Omega \nu_E(x) \cdot \boldsymbol{\varphi}\, d|V_{E_k}|(x) = \int_{\Omega\times\mathbb{S}^{d-1}} s \cdot \boldsymbol{\varphi}(x)\, dV(x,s). \end{aligned}$$

Hence V can be used to describe the limit of $H_{\chi_{E_k}}$ as well as the limit of $\nabla \chi_{E_k}$.

Now we define a varifold solution of the two-phase flow as follows:

Definition 2.4. (Varifold solutions)
Let $\sigma > 0$. Then

$$\begin{aligned}
&\mathbf{v} \in L^\infty(0,\infty; L^2_\sigma(\mathbb{R}^d)) \cap L^2(0,\infty; H^1_0(\mathbb{R}^d)^d),\\
&\chi \in L^\infty(0,\infty; BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d \times (0,\infty); \{0,1\})),\\
&V \in L^\infty_{\omega*}(0,\infty; \mathcal{M}(\Omega \times \mathbb{S}^{d-1}))
\end{aligned}$$

are called a varifold solution of the two-phase flow *for initial data* $\mathbf{v}_0 \in L^2_\sigma(\mathbb{R}^d)$ *and* $\chi_0 = \chi_{\Omega_0^+}$ *for a bounded domain* $\Omega_0^+ \subset\subset \mathbb{R}^d$ *of finite perimeter if the following conditions are satisfied:*

(*i*) *(2.8) holds for all* $\boldsymbol{\varphi} \in C^\infty_{(0)}(\mathbb{R}^d \times [0,\infty))^d$ *with* $\operatorname{div} \boldsymbol{\varphi} = 0$, *where* $\langle H_{\Gamma(t)}, \boldsymbol{\varphi}\rangle$ *is replaced by*

$$\langle \delta V(t), \boldsymbol{\varphi}\rangle = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \operatorname{tr}((I - s\otimes s)\nabla \boldsymbol{\varphi}(x))\, dV(s,x), \quad \boldsymbol{\varphi} \in C^1_0(\Omega)^d.$$

(*ii*) *The modified energy inequality*

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{v}(t)\|_2^2 + \sigma \|V(t)\|_{\mathcal{M}(\Omega\times\mathbb{S}^{d-1})}\\
&+ \int_0^t \int_{\mathbb{R}^d} 2\eta(\chi)|D\mathbf{v}|^2\, dx\, d\tau \le \frac{1}{2}\|\mathbf{v}_0\|_2^2 + \sigma\|\nabla\chi_0\|_{\mathcal{M}(\Omega)} \qquad (2.21)
\end{aligned}$$

holds for almost all $t \in (0,\infty)$.

(*iii*) *The compatibility condition*

$$-\langle \nabla\chi(t), \varphi\rangle = \int_{\Omega\times\mathbb{S}^{d-1}} s\cdot \boldsymbol{\varphi}(x)\, dV(x,s), \quad \boldsymbol{\varphi} \in C_0(\Omega)^d, \qquad (2.22)$$

holds for almost all $t > 0$.

Here $L^\infty_{\omega*}(0,T;X')$ denotes the space of weakly-$*$ measurable essentially bounded functions $f\colon (0,T) \to X'$.

Remark 2.5. (*i*) *Let* $(V_x(t), |V(t)|)$, $x \in \mathbb{R}^d$, *denote the disintegration of* $V(t) \in \mathcal{M}(\mathbb{R}^d \times \mathbb{S}^{d-1})$ *as described above. Then (2.22) implies that* $|\nabla\chi(t)|(A) \le |V(t)|(A)$ *for all open sets* A *and almost all* $t \in (0,\infty)$. *Hence* $|\nabla\chi(t)|$ *is absolutely continuous with respect to* $|V(t)|$ *and*

$$\int_{\mathbb{R}^d} f(x)\, d|\nabla\chi(t)| = \int_{\mathbb{R}^d} f(x)\alpha_t(x)\, d|V(t)|, \quad f \in C_0(\mathbb{R}^d),$$

for a $|V(t)|$*-measurable function* $\alpha_t\colon \mathbb{R}^d \to [0,\infty)$ *with* $|\alpha_t(x)| \le 1$ *almost everywhere. In particular, this implies* $\operatorname{supp}\nabla\chi_t \subseteq \operatorname{supp} V(t)$ *and* $\|\nabla\chi(t)\|_{\mathcal{M}} \le \|V(t)\|_{\mathcal{M}}$ *for almost all* $t \in (0,\infty)$. *Hence every* *varifold solution satisfies the energy inequality (2.17) for almost every* $t > 0$.

Moreover, if $E(t) = \{x \in \mathbb{R}^d : \chi(x,t) = 1\}$, $t > 0$, *then (2.22) yields the relation*

$$\int_{\mathbb{S}^{d-1}} s\, dV_x(t)(s) = \begin{cases} \alpha_t(x)\boldsymbol{\nu}_{E(t)}(x) & \text{if } x \in \partial^* E_t \\ 0 & \text{else} \end{cases}$$

for $|V(t)|$*-almost every* $x \in \mathbb{R}^d$ *and almost every* $t > 0$. *– In other words, the expectation of* $V_x(t)$ *is proportional to the normal* $\boldsymbol{\nu}$ *on the interface described by* $\nabla\chi$ *and zero away from it.*

(ii) *In general, it is an open problem whether* $V(t)$ *is a so-called countably* $(d-1)$*-rectifiable varifold, which implies that up to orientation* $V_x(t)$ *is a Dirac measure for* $|V(t)|$*-almost every* x. *Then* $V(t)$ *can naturally be identified with a countably* $(d-1)$*-rectifiable set – a "surface" – equipped with a density* $\theta_t \geq 0$. *So far only a sufficient condition for the rectifiability of* $V(t)$ *in terms of the first variation* $\delta V(t)$ *is known, cf. [2, Section 4].*

(iii) *As noted above, the existence of weak solutions to the two-phase flow with surface tension is open. But a general property of varifold solutions is that a varifold solution is a weak solution if the energy equality holds, i.e., (2.17) holds with equality for almost every* $t > 0$. *See [3, Proposition 1.5] for details.*

Theorem 2.6. (Existence of Varifold Solutions)
Let $\sigma > 0$, $d = 2, 3$. *Then for every* $\mathbf{v}_0 \in L^2_\sigma(\mathbb{R}^d)$ *and* $\chi_0 = \chi_{\Omega_0^+}$ *where* $\Omega_0^+ \subset\subset \mathbb{R}^d$ *is a bounded* C^1*-domain there is a varifolds solution of the two-phase flow with surface tension* $\sigma > 0$ *in the sense of Definition 2.4.*

We refer to [3, Theorem 1.6] for further properties, which can be shown for the constructed varifold.

Further and related results: The result was extend by Yeressian [80], where the existence of axisymmetric varifold solutions in the case of axisymmetric initial values in $\mathbb{R}^3$ was shown. In the case $d = 2$ and $\rho_\pm = \eta_\pm = 1$ existence of varifold solutions was also obtained by Ambrose et al. [22]. Their definitions and statements are slightly different; but the result is essentially the same. Moreover, they discuss possible defects in the surface tension functional. Earlier generalized solutions for the two-phase flow with surface tension were also constructed by Salvi [67]. But in the latter work the meaning of the mean curvature functional is not specified and can be chosen arbitrarily within in a certain function space. Moreover, we note that a Bernoulli free boundary problem with surface tension was discussed by Wagner [79]. Finally, we note that existence of varifold solutions were also obtained in [14] by a sharp interface limit of a diffuse interface model, which will be discussed in the next section. But the definition of varifold solution and their properties are slightly different. We note that the limit system obtained in this sharp interface limit depends on the scaling of a mobility coefficient in the diffuse interface model. In one case the classical model (2.1)-(2.7) is obtained in another case the system studied in the next section is obtained.

2.3 Existence of Weak Solutions for a Navier–Stokes/Mullins–Sekerka System

An alternative model to the classical two-phase flow model (2.1)-(2.7) is the following Navier–Stokes/Mullins–Sekerka system:

$$
\begin{aligned}
\rho_\pm \partial_t \mathbf{v} + \rho_\pm \mathbf{v}\cdot\nabla\mathbf{v} - \eta_\pm \Delta\mathbf{v} + \nabla p &= 0 && \text{in } \Omega_\pm(t), && (2.23)\\
\operatorname{div}\mathbf{v} &= 0 && \text{in } \Omega_\pm(t), && (2.24)\\
\Delta\mu &= 0 && \text{in } \Omega_\pm(t), && (2.25)\\
\mu|_{\Gamma(t)} &= \sigma H && \text{on } \Gamma(t), && (2.26)\\
[\mathbf{v}]_-^+ &= 0 && \text{on } \Gamma(t), && (2.27)\\
-[2\eta D\mathbf{v}]_-^+ \boldsymbol{\nu} + [p]_-^+ &= \sigma H \boldsymbol{\nu} && \text{on } \Gamma(t), && (2.28)\\
\boldsymbol{\nu}\cdot\mathbf{v} - \mathcal{V} &= m[\boldsymbol{\nu}\cdot\nabla\mu]_-^+ && \text{on } \Gamma(t), && (2.29)\\
\mathbf{v} &= 0 && \text{on } \partial\Omega, && (2.30)\\
\boldsymbol{\nu}_{\partial\Omega}\cdot\nabla\mu|_{\partial\Omega} &= 0 && \text{on } \partial\Omega, && (2.31)\\
\mathbf{v}|_{t=0} &= \mathbf{v}_0 && \text{in } \Omega && (2.32)
\end{aligned}
$$

for $t > 0$ together with $\Omega_+(0) = \Omega_0^+$. The system arises naturally as a sharp interface limit of the diffuse interface models discussed in Section 3.1 if the mobility coefficient m does not vanish in the limit. If $m = 0$ in the system above, then the equations for μ decouple from the rest of the system and can be deleted from the system. Then the system coincides with the classical model (2.1)-(2.7). Here $\mu\colon \Omega\times(0,\infty)\to\mathbb{R}$ is a new quantity in the system and plays the role of a chemical potential associated to a free energy, which is $\sigma\mathcal{H}^{d-1}$ restricted to the interface $\Gamma(t)$. Moreover, $m > 0$ is a mobility coefficient, which influences the strength of a (non-local) diffusion in the system. We note that (2.25), (2.26), (2.29), and (2.31) for $\mathbf{v} = 0$ is the so-called Mullins–Sekerka system (or two-phase Hele-Shaw system), which arises as sharp interface limit of the Cahn-Hilliard equation, which models phase separation in a two-component mixture. It is well-known that solutions of this system show the so-called *Ostwald ripening effect* in the long-time dynamics, which is the diffusion of mass from smaller droplets to larger droplets until finally one large droplet remains. This effect is also present in the full system (2.23)-(2.32).

In the following we will discuss a result on existence of weak solutions for the Navier–Stokes/Mullins–Sekerka system above. To this end we note that sufficiently smooth solutions of (2.1)-(2.7) satisfy the following energy dissipation identity,

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_\Omega |\mathbf{v}(t)|^2\,dx + \sigma\frac{d}{dt}\mathcal{H}^{d-1}(\Gamma(t))\\&= -\int_\Omega 2\eta(\chi)|D\mathbf{v}|^2\,dx - m\int_\Omega |\nabla\mu|^2\,dx,\end{aligned} \tag{2.33}$$

where $\eta(0) = \eta_-$ and $\eta(1) = \eta_+$ as before. This identity can be verified by multiplying (2.23) and (2.25) with $\mathbf{v}$, μ, resp., integrating and using the boundary and interface conditions (2.25)-(2.31). This energy equality motivates the choice of solution spaces in our weak formulation and shows that the regularization introduced for $m > 0$ yields an additional dissipation term. In particular, we expect $\mu(\cdot,t) \in H^1(\Omega)$ for almost all $t \in \mathbb{R}_+$ and formally, using Sobolev inequality and (2.6), that $H(\cdot,t) \in L^4(\Gamma(t))$ for $d \leq 3$. This gives some indication of extra regularity properties of the interface in the model with $m > 0$ and is in big contrast to the classical model (the case $m = 0$), where no control of the mean curvature of $\Gamma(t)$ can be derived from the energy identity in a straight forward manner.

The following result on existence of weak solutions of (2.23)-(2.32) was proved in [16].

Theorem 2.7. *Let $d = 2,3$, $T > 0$, let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary or let $\Omega = \mathbb{T}^d$, let $\eta(0) := \eta_-$, $\eta(1) := \eta_+$ and $\sigma, m > 0$. Then for any $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $\chi_0 \in BV(\Omega;\{0,1\})$ there are*

$$\begin{aligned}&\mathbf{v} \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_0(\Omega)^d),\\&\chi \in L^\infty_{w*}(0,T;BV(\Omega;\{0,1\})),\\&\mu \in L^2(0,T;H^1(\Omega)),\end{aligned}$$

that satisfy (2.1)-(2.7) in the following sense: For almost all $t \in (0,T)$ the phase interface $\partial^*\{\chi(\cdot,t) = 1\}$ has a generalized mean curvature vector $\mathbf{H}(t) \in L^s(d|\nabla\chi(t)|)^d$ with $s = 4$ if $d = 3$ and $1 \leq s < \infty$ arbitrary if $d = 2$, such that

$$\begin{aligned}&\int_0^T\int_\Omega \left(-\mathbf{v}\cdot\partial_t\boldsymbol{\varphi} + (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\varphi} + \eta(\chi)D\mathbf{v} : D\boldsymbol{\varphi}\right)dx\,dt\\&\quad -\int_\Omega \boldsymbol{\varphi}|_{t=0}\cdot\mathbf{v}_0\,dx = \sigma\int_0^T\int_\Omega \mathbf{H}(t)\cdot\boldsymbol{\varphi}(t)\,d|\nabla\chi(t)|\,dt\end{aligned} \tag{2.34}$$

holds for all $\boldsymbol{\varphi} \in C^\infty([0,T];C^\infty_{0,\sigma}(\Omega))$ with $\boldsymbol{\varphi}|_{t=T} = 0$,

$$\begin{aligned}&\int_0^T\int_\Omega \chi(\partial_t\psi + \operatorname{div}(\psi\mathbf{v}))\,dx\,dt + \int_\Omega \chi_0(x)\psi(0,x)\,dx\\&= m\int_0^T\int_\Omega \nabla\mu\cdot\nabla\psi\,dx\,dt\end{aligned} \tag{2.35}$$

holds for all $\psi \in C^\infty([0,T] \times \overline{\Omega})$ *with* $\psi|_{t=T} = 0$ *and*

$$\sigma \mathbf{H}(t,.) = \mu(t,.) \frac{\nabla \chi(\cdot,t)}{|\nabla \chi|(\cdot,t)} \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial^* \{\chi(t,.) = 1\} \tag{2.36}$$

holds for almost all $0 < t < T$.

Here the concept of generalized mean curvature for non-smooth phase interfaces is taken from [65] and can also be found in [16, Definition 4.4].

Remark 2.8. (2.34) *is the weak formulation of* (2.23), (2.28), *and* (2.32). *It is obtained from testing* (2.23) *with* $\boldsymbol{\varphi}$ *in* $\Omega_\pm(t)$, *integrating over* $\Omega_+(t) \cup \Omega_-(t)$ *and using Gauss' theorem,* (2.28), *and* (2.32). *Moreover,* (2.35) *is a weak formulation of* (2.25), (2.29), (2.31), *and* $\Omega_+(0) = \Omega_0^+$. *The conditions* (2.24), (2.27) *and* (2.30) *are included in the choice of the function spaces, namely* $\mathbf{v}(t) \in H_0^1(\Omega)$ *for almost every* $t \in (0,T)$, *and* (2.26) *is formulated in* (2.36).

The proof is essentially based on a compactness result of Schätzle [68] for $(d-1)$-dimensional hypersurfaces with mean curvature given as the trace of an ambient Sobolev function in $W^1_p(\mathbb{R}^d)$ for $p > \frac{d}{2}$. For the application of this result the bound of $\nabla\mu \in L^2(0,T;L^2(\Omega))^d$ obtained from (2.33) is used. Such a control of the curvature of the interface is missing for the classical model (2.1)-(2.7), which is one of the main reasons that existence of weak solutions to the latter system is open in general if $\sigma > 0$.

3 Diffuse Interface Models

In diffuse interface models a partial mixing of the two incompressible fluids in a thin interfacial region is assumed. In the following two fluids with mass densities ρ_- and ρ_+ are considered. The mass balance equation for the two fluids in local form is given by

$$\partial_t \rho_\pm + \operatorname{div} \widehat{\mathbf{J}}_\pm = 0$$

where $\widehat{\mathbf{J}}_\pm$ are the mass fluxes of the fluids $+$ and $-$. Introducing the velocities $\mathbf{v}_\pm = \widehat{\mathbf{J}}_\pm / \rho_\pm$ we can rewrite the mass balance as

$$\partial_t \rho_\pm + \operatorname{div}(\rho_\pm \mathbf{v}_\pm) = 0\,. \tag{3.1}$$

The further modeling now crucially depends on the way how an averaged velocity $\mathbf{v}$ is defined. Precise choices of $\mathbf{v}$ will be given below.

The mass flux of the two fluids relative to the velocity $\mathbf{v}$ is denoted by

$$\mathbf{J}_\pm = \widehat{\mathbf{J}}_\pm - \rho_\pm \mathbf{v}$$

and the mass balances are rewritten as

$$\partial_t \rho_\pm + \operatorname{div}(\rho_\pm \mathbf{v}) + \operatorname{div} \mathbf{J}_\pm = 0 \tag{3.2}$$

where $\mathbf{J}_\pm$ are diffusive flow rates. Defining the total mass

$$\rho = \rho_+ + \rho_-$$

one obtains

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) + \operatorname{div}(\mathbf{J}_+ + \mathbf{J}_-) = 0\,. \tag{3.3}$$

One observes that the classical continuity equation does not hold if $\operatorname{div}(\mathbf{J}_+ + \mathbf{J}_-) \neq 0$.

Considering a conservation of linear momentum with respect to the above velocity we obtain

$$\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div} \widetilde{\mathbf{T}} \tag{3.4}$$

where $\widetilde{\mathbf{T}}$ is the stress tensor which has to be specified by constitutive assumptions. It turns out, see [20], that $\widetilde{\mathbf{T}}$ in general is not an objective tensor. Rewriting (3.4) with the help of the mass conservation (3.3) one gets with $\widetilde{\mathbf{J}} = \mathbf{J}_1 + \mathbf{J}_2$

$$\begin{aligned} \rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) &= \operatorname{div} \widetilde{\mathbf{T}} + (\operatorname{div} \widetilde{\mathbf{J}}) \cdot \mathbf{v} \\ &= \operatorname{div}(\widetilde{\mathbf{T}} + \mathbf{v} \otimes \widetilde{\mathbf{J}}) - (\widetilde{\mathbf{J}} \cdot \nabla)\mathbf{v}\,. \end{aligned}$$

The system now allows for an objective tensor

$$\mathbf{T} = \widetilde{\mathbf{T}} + \mathbf{v} \otimes \widetilde{\mathbf{J}} \tag{3.5}$$

and we obtain

$$\rho \partial_t \mathbf{v} + \Big((\rho \mathbf{v} + \widetilde{\mathbf{J}}) \cdot \nabla \Big) \mathbf{v} = \operatorname{div} \mathbf{T} \tag{3.6}$$

which is the classical formulation if $\widetilde{\mathbf{J}} = 0$, which is equivalent to $\mathbf{J}_+ = -\mathbf{J}_-$. The work [12] and [20] give more details concerning the objectivity of the mass-momentum system with $\widetilde{\mathbf{J}} \neq 0$.

It remains to specify the averaged velocity $\mathbf{v}$ with the help of the individual velocities $\mathbf{v}_-$ and $\mathbf{v}_+$. Two choices are used in the literature.

The *volume averaged velocity*

$$\mathbf{v} = u_- \mathbf{v}_- + u_+ \mathbf{v}_+$$

where u_- and u_+ are the volume fractions of the two fluids and the *mass averaged-velocity*

$$\mathbf{v} = \frac{\rho_1}{\rho} \mathbf{v}_1 + \frac{\rho_2}{\rho} \mathbf{v}_2\,.$$

In the following we discuss the two modeling variants which result from different choices of the averaged velocity $\mathbf{v}$.

3.1 Models Based on the Volume Averaged Velocity

In the interfacial zone the total volume occupied by each fluid is no longer conserved. Insisting on a conservation of volume during the mixing process would lead to the necessity that when fluid + flows out of a region an amount of fluid − of the same volume would have to enter this region. Defining the specific (constant) density of the unmixed fluid by $\tilde{\rho}_\pm$ we introduce the volume fraction

$$u_\pm = \rho_\pm/\tilde{\rho}_\pm \tag{3.7}$$

and the above discussion on the volume conservation leads to

$$u_1 + u_2 = 1 \tag{3.8}$$

which states that the excess volume is zero. Multiplying (3.1) with $1/\tilde{\rho}_\pm$, using $u_1 + u_2 = 1$ and the definition of $\mathbf{v}$ as the volume averaged velocity gives

$$\begin{aligned} 0 &= \partial_t\left(\frac{\rho_+}{\tilde{\rho}_+} + \frac{\rho_-}{\tilde{\rho}_-}\right) + \operatorname{div}\left(\frac{\rho_+}{\tilde{\rho}_+}\mathbf{v}_+ + \frac{\rho_-}{\tilde{\rho}_-}\mathbf{v}_-\right) && (3.9)\\ &= \partial_t(u_- + u_+) + \operatorname{div}\mathbf{v} && (3.10)\\ &= \operatorname{div}\mathbf{v}\,. && (3.11) \end{aligned}$$

From (3.2) one derives

$$\partial_t u_\pm + \operatorname{div}(u_\pm \mathbf{v}) + \operatorname{div}\widetilde{\mathbf{J}}_\pm = 0$$

where we set $\widetilde{\mathbf{J}}_\pm = \mathbf{J}_\pm/\tilde{\rho}_\pm$. Taking the difference of these two equations gives for $\varphi = u_+ - u_-$ the equation

$$\partial_t\varphi + \operatorname{div}(\rho\mathbf{v}) + \operatorname{div}\mathbf{J}_\varphi = 0 \tag{3.12}$$

where

$$\mathbf{J}_\varphi = \mathbf{J}_+/\tilde{\rho}_+ - \mathbf{J}_-/\tilde{\rho}_-\,.$$

We also note that (3.7) and (3.8) together with $\rho = \rho_+ + \rho_-$ gives

$$\rho = \rho(\varphi) = \tilde{\rho}_+\frac{1+\varphi}{2} + \tilde{\rho}_-\frac{1-\varphi}{2}\,,$$

i.e., ρ is an affine linear function of φ. Using $\rho'(\varphi) = (\tilde{\rho}_+ - \tilde{\rho}_-)/2$ we obtain from (3.12)

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{v}) + \operatorname{div}\widetilde{\mathbf{J}} = 0 \tag{3.13}$$

where the relation $(\tilde{\rho}_+ - \tilde{\rho}_-)\mathbf{J}_\varphi = 2\,\widetilde{\mathbf{J}}$ holds.

Motivated by the discussion in the introduction we introduce a total energy density

$$e(\mathbf{v},\varphi,\nabla\varphi) = \frac{\rho}{2}|\mathbf{v}|^2 + f(\varphi,\nabla\varphi)$$

as the sum of a kinetic and a free energy. As an example one can take $f(\varphi, \nabla\varphi) = \hat{\sigma}\left(\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\right)$, cf. (1.1). In an isothermal situation the appropriate formulation of the second law of thermodynamics is given as the following dissipation inequality, see, e.g., [49], as follows

$$\frac{d}{dt}\int_{V(t)} e(\mathbf{v}, \varphi, \nabla\varphi)dx + \int_{\partial V(t)} \mathbf{J}_e \cdot \boldsymbol{\nu}\, d\mathcal{H}^{n-1} \leq 0$$

where $V(t)$ is a test volume which is transported with the flow, described by $\mathbf{v}$, and $\mathbf{J}_e$ is a general energy flux which will be specified later. Using a transport theorem and the fact that the test volume is arbitrary we obtain the local form, see [13], [56],

$$-\mathcal{D} := \partial_t e + \operatorname{div}(\mathbf{v}e) + \operatorname{div}\mathbf{J}_e \leq 0\,. \tag{3.14}$$

One can now use the Lagrange multiplier method of Liu and Müller [56, 61] to derive constitutive relations which guarantee that the second law is fulfilled. Every fields $(\varphi, \mathbf{v})$ which fulfill the dissipation inequality (3.14), (3.9) and (3.12) also fulfill

$$-\mathcal{D} = \partial_t e + \mathbf{v}\cdot\nabla\varphi - \mu(\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi + \operatorname{div}\mathbf{J}_\varphi) \leq 0 \tag{3.15}$$

where μ is a Lagrange multiplier which will be specified later.

Using (3.6), (3.13) we obtain

$$\begin{aligned}
\partial_t\left(\frac{\rho}{2}|\mathbf{v}|^2\right) + \operatorname{div}\left(\frac{\rho}{2}|\mathbf{v}|^2\mathbf{v}\right) &= -\frac{|\mathbf{v}|^2}{2}\operatorname{div}\widetilde{\mathbf{J}} + (\operatorname{div}\mathbf{T} - (\widetilde{\mathbf{J}}\cdot\nabla)\mathbf{v})\cdot\mathbf{v} \\
&= \operatorname{div}\left(-\frac{1}{2}|\mathbf{v}|^2\widetilde{\mathbf{J}} + \mathbf{T}^T\mathbf{v}\right) - \mathbf{T} : \nabla\mathbf{v}\,.
\end{aligned}$$

Denoting by $f_{,\varphi}$ and $f_{,\nabla\varphi}$ the partial derivatives with respect to φ and $\nabla\varphi$ we obtain

$$D_t f = f_{,\varphi} D_t\varphi + f_{,\nabla\varphi}\cdot D_t\nabla\varphi$$

where

$$D_t u = \partial_t u + \mathbf{v} - \nabla u$$

is the material derivative. Using

$$D_t\nabla\varphi = \nabla D_t\varphi - (\nabla\mathbf{v})^T\nabla\varphi \tag{3.16}$$

yields that (3.15) gives after some computations

$$\begin{aligned}
-\mathcal{D} \;=\; & \nabla\cdot\left(\mathbf{J}_e - \widetilde{\mathbf{J}}\frac{|\mathbf{v}|^2}{2} + \mathbf{T}^T\mathbf{v} - \mu\mathbf{J}_\varphi + f_{,\nabla\varphi}D_t\varphi\right) \\
& + (f_{,\varphi} - \mu - \operatorname{div} f_{,\nabla\varphi})D_t\varphi \\
& - (\mathbf{T} + \nabla\varphi\otimes f_{,\nabla\varphi}) : \nabla\mathbf{v} + \nabla\mu\cdot\mathbf{J}_\varphi \leq 0\,.
\end{aligned}$$

Choosing the chemical potential as

$$\mu = f_{,\varphi} - \operatorname{div} f_{,\nabla\varphi}$$

and

$$\mathbf{J}_e = \widetilde{\mathbf{J}}\frac{|\mathbf{v}|^2}{2} + \mathbf{T}^T\mathbf{v} - \mu\mathbf{J}_\varphi + f_{,\nabla\varphi}D_t\varphi$$

we end up with the dissipation inequality

$$(\mathbf{T} + \nabla\varphi \otimes f_{,\nabla\varphi}) : \nabla\mathbf{v} - \nabla\mu \cdot \mathbf{J}_\varphi \geq 0\,.$$

Often it is convenient, see, e.g., [49], to introduce an extra stress $\mathbf{S}$ and the pressure p such that

$$\widetilde{\mathbf{S}} = \mathbf{T} + p\ \mathrm{Id}\ .$$

Due to the incompressible condition the pressure p is still indeterminate, see also [49]. With the stress $\widetilde{\mathbf{S}}$ we obtain

$$(\widetilde{\mathbf{S}} + \nabla\varphi \otimes f_{,\nabla\varphi}) : \nabla\mathbf{v} - \nabla\mu \cdot \mathbf{J}_\varphi \geq 0$$

since $\operatorname{div}\mathbf{v} = 0$. The term $\mathbf{S} = \widetilde{\mathbf{S}} + \nabla\varphi \otimes f_{,\nabla\varphi}$ is the viscous stress tensor since it corresponds to irreversible changes of energy due to friction.

We now consider specific constitutive assumptions. For a classical Newtonian fluid one choose

$$\mathbf{S} = \widetilde{\mathbf{S}} + \nabla\varphi \otimes f_{,\nabla\varphi} = 2\eta(\varphi)D\mathbf{v}$$

for some φ-dependent viscosity $\eta(\varphi) \geq 0$. The simplest form of the flux $\mathbf{J}_\varphi$ is of Fick's type

$$\mathbf{J} = -m(\varphi)\nabla\mu$$

where $m(\varphi) \geq 0$ in order to guarantee that the dissipation inequality is fulfilled. Choosing

$$f(\varphi, \nabla\varphi) = \hat{\sigma}\left(\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\right)$$

gives in conclusion the following diffuse interface model

$$\begin{aligned}
\rho\partial_t\mathbf{v} + ((\rho\mathbf{v} + \widetilde{\mathbf{J}})\cdot\nabla)\mathbf{v} - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \nabla p &= -\hat{\sigma}\varepsilon\operatorname{div}(\nabla\varphi\otimes\nabla\varphi)\,, && (3.17)\\
\operatorname{div}\mathbf{v} &= 0\,, && (3.18)\\
\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi &= \operatorname{div}(m(\varphi)\nabla\mu)\,, && (3.19)\\
\frac{\hat{\sigma}}{\varepsilon}\psi'(\varphi) - \hat{\sigma}\varepsilon\Delta\varphi &= \mu\,. && (3.20)
\end{aligned}$$

We remark that

$$\widetilde{\mathbf{J}} = \frac{\tilde{\rho}_+ - \tilde{\rho}_-}{2}\mathbf{J}_\varphi = -\frac{\tilde{\rho}_+ - \tilde{\rho}_-}{2}m(\varphi)\nabla\mu$$

which gives that the term involving $\widetilde{\mathbf{J}}$ in the momentum equation vanishes for equal densities, i.e., if $\tilde{\rho}_+ = \tilde{\rho}_-$. In the case of equal densities one hence recovers the famous "Model H" discussed in Hohenberg and Halperin [50]. The model (3.17)-(3.20) was first derived in [13]. However, other diffuse interface models based on a volume averaged velocity were also studied in [31, 41]. For both models neither global nor local energy inequalities seem to be known. The model of Ding, Spelt and Shu [41] is given by (3.17)-(3.20) with $\widetilde{\mathbf{J}}$ being zero which hence drops a term which is important for the dissipation inequality.

Using the fact that the pressure can be redefined there are a few reformulations of (3.17) which are convenient. Due to the identity

$$\mu\nabla\varphi = \nabla(f(\varphi,\nabla\varphi)) - \operatorname{div}(\nabla\varphi\otimes f_{,\nabla\varphi})$$

it is possible to redefine the pressure as follows

$$\hat{p} = p - f(\varphi,\nabla\varphi)$$

and we obtain instead of (3.17)

$$\rho\partial_t\mathbf{v} + ((\rho\mathbf{v}+\widetilde{\mathbf{J}})\cdot\nabla)\mathbf{v} - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \nabla p = \mu\nabla\varphi\,. \tag{3.21}$$

For the following we assume that $f(\varphi,\nabla\varphi) = \hat{\sigma}\left(\varepsilon\frac{|\nabla\varphi|^2}{2} + \frac{\psi(\varphi)}{\varepsilon}\right)$, cf. (1.1). In some situations it is more convenient to consider the formulation

$$\begin{aligned}\rho\partial_t\mathbf{v} + ((\rho\mathbf{v}+\widetilde{\mathbf{J}})\cdot\nabla)\mathbf{v} - \nabla\cdot(\eta(\varphi)D(\mathbf{v})) + \nabla p\\ = \nabla\cdot\left(\hat{\sigma}\varepsilon|\nabla\varphi|^2\left(\operatorname{Id}-\frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)\right).\end{aligned}$$

It turns out that

$$\varepsilon|\nabla\varphi|^2\left(\operatorname{Id}-\frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)$$

in some sense converges in the sharp interface limit $\varepsilon\to 0$ to a multiple of

$$(\operatorname{Id}-\boldsymbol{\nu}\otimes\boldsymbol{\nu})\delta_\Gamma$$

where δ_Γ is a surface Dirac distribution concentrated on the interface and $\operatorname{Id}-\boldsymbol{\nu}\otimes\boldsymbol{\nu}$ is the projection onto the interface which is up to a factor the relevant surface stress tensor, cf. (2.16) below.

Moreover, using (3.13) one obtains that (3.17) is equivalent to

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes(\rho\mathbf{v}+\widetilde{\mathbf{J}})) - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \nabla p = -\hat{\sigma}\varepsilon\operatorname{div}(\nabla\varphi\otimes\nabla\varphi). \tag{3.22}$$

Furthermore, in the same way as discussed above the right-hand side of (3.22) can be replaced by $\mu\nabla\varphi$ if p is replaced by $p+\varepsilon\frac{|\nabla\varphi|^2}{2}+\frac{\psi(\varphi)}{\varepsilon}$ and for the new pressure we obtain

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes(\rho\mathbf{v}+\widetilde{\mathbf{J}})) - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \nabla p = \mu\nabla\varphi\,. \tag{3.23}$$

3.2 Model Based on the Mass Averaged Velocity

A model based on a mass averaged velocity was derived by Lowengrub and Truskinovsky [58]. They define the averaged velocity $\mathbf{v}$ as

$$\mathbf{v} = \frac{\rho_-\mathbf{v}_- + \rho_+\mathbf{v}_+}{\rho} \,.$$

In this case the mass balance becomes

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 \,, \tag{3.24}$$

which is obtained by adding both mass balances in (3.1). Defining the mass concentrations

$$c_\pm = \frac{\rho_\pm}{\rho}$$

we now introduce the concentration difference

$$c = c_+ - c_- \,,$$

as phase field variable. We want to model the mixing of two incompressible fluids and in the following assume that the total density ρ depends only on the concentration difference c. Hence we assume that there exists a function $\hat{\rho} : [-1, 1] \to (0, \infty)$ such that $\rho = \hat{\rho}(c)$. Adapting a model of a simple mixture, see [51], one obtains

$$\frac{\rho_+}{\tilde{\rho}_+} + \frac{\rho_-}{\tilde{\rho}_-} = 1 \,.$$

This condition is just the assumption of zero excess volume which was discussed earlier. In this case the functional dependence between ρ and $c = c_+ - c_-$ is given as (one has to use $c_+ + c_- = 1$)

$$\hat{\rho}(c) = \left(\frac{1}{2}(1 + c)/\tilde{\rho}_+ + \frac{1}{2}(1 - c)/\tilde{\rho}_- \right)^{-1} \,. \tag{3.25}$$

However, in what follows we allow for a more general relation $\rho = \hat{\rho}(c)$. Taking the difference of the mass balances (3.2) now yields (using $\rho_\pm = \hat{\rho}(c)c_\pm$ and $c = c_+ - c_-$)

$$\partial_t(\rho c) + \operatorname{div}(\rho c \, \mathbf{v}) + \operatorname{div} \mathbf{j} = 0 \tag{3.26}$$

which is, using (3.24), equivalent to

$$\rho(\partial_t c + \mathbf{v} \cdot \nabla c) + \operatorname{div} \mathbf{j} = 0 \tag{3.27}$$

where $\mathbf{j} = \mathbf{J}_+ - \mathbf{J}_-$. The equation (3.27) has to be supplemented with the momentum equation (3.4) which, using (3.24), can be rewritten as

$$\rho(c)(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = \operatorname{div} \widetilde{\mathbf{T}} \,.$$

As in Section 3.1 we require a free energy inequality

$$-\mathcal{D} := \partial_t e + \operatorname{div}(e\mathbf{v}) + \nabla \cdot \mathbf{j}_e \leq 0$$

where

$$e(\mathbf{v}, c, \nabla c) = \frac{\hat{\rho}(c)}{2}|\mathbf{v}|^2 + \hat{\rho}(c)\hat{f}(c, \nabla c)\,.$$

It turns out that for the mass averaged velocity it is more convenient to work with a free energy density $\hat{f}$ per unit mass. For solutions of the mass and momentum equations we obtain, using Lagrange multipliers λ_ρ and μ,

$$\partial_t e + \operatorname{div}(\mathbf{v}e) + \operatorname{div}\mathbf{j}_e - \lambda_\rho(D_t\rho + \rho\operatorname{div}\mathbf{v}) - \mu(\rho D_t c + \operatorname{div}\mathbf{j}) - \mathbf{v}\cdot(\rho D_t\mathbf{v} - \operatorname{div}\widetilde{\mathbf{T}}) \leq 0$$

which is equivalent to

$$\begin{aligned} &\rho\hat{f}_{,c}D_t c + \rho\hat{f}_{,\nabla c}\cdot(D_t\nabla c) + \operatorname{div}\mathbf{j}_e - \lambda_\rho D_t\rho - \lambda_\rho\rho\operatorname{div}\mathbf{v} \\ &-\mu\rho D_t c - \operatorname{div}(\mu\mathbf{j}) + \nabla\mu\cdot\mathbf{j} + \operatorname{div}(\widetilde{\mathbf{T}}^T\mathbf{v}) - \widetilde{\mathbf{T}} : \nabla\mathbf{v} \leq 0\,. \end{aligned}$$

Using the identity

$$\rho f_{,\nabla c}\cdot(D_t\nabla c) = \operatorname{div}(\rho f_{,\nabla c}D_t c) - (\operatorname{div}(\rho f_{,\nabla c}))D_t c - \rho(\nabla\mathbf{v}) : (\nabla c \otimes f_{,\nabla c}),$$

which follows using (3.16), we obtain

$$\begin{aligned} &D_t c\,(\rho f_{,c} - \operatorname{div}(\rho f_{,\nabla c}) - \lambda_\rho\hat{\rho}'(c) - \mu\rho) \\ &\quad +\nabla\mathbf{v} : (-\lambda_\rho\rho\,\mathrm{Id} - \rho\nabla c\otimes f_{,\nabla c} - \widetilde{\mathbf{T}}) \\ &+\nabla\mu\cdot\mathbf{j} + \operatorname{div}(\mathbf{j}_e + \rho f_{,\nabla c}D_t c - \mu\mathbf{j} - \widetilde{\mathbf{T}}^T\mathbf{v}) \leq 0\,. \end{aligned}$$

This is true for all solutions of the mass and momentum balance equations if

$$\begin{aligned} \mu &= \frac{1}{\rho}(-\operatorname{div}(\rho f_{,\nabla c}) + \rho f_{,c} + \lambda_\rho\hat{\rho}'(c)), \\ \mathbf{j}_e &= \mu\mathbf{j} - f_{,\nabla c}D_t c + \widetilde{\mathbf{T}}^T\mathbf{v}, \end{aligned}$$

and

$$\nabla\mathbf{v} : \mathbf{S} - \nabla\mu\cdot\mathbf{j} \geq 0\,,$$

where

$$\mathbf{S} = \widetilde{\mathbf{T}} + \lambda_\rho\rho\,\mathrm{Id} + \rho\nabla c\otimes f_{,\nabla c}\,.$$

Interpreting $\lambda_\rho\rho$ as the pressure, i.e., setting

$$\lambda_\rho = -\frac{p}{\rho}$$

and making the specific choices

$$\begin{aligned} \mathbf{S} &= \eta(c)(\nabla\mathbf{v} + \nabla\mathbf{v}^T) + \lambda(c)\operatorname{div}\mathbf{v}\,\mathrm{Id} \\ \mathbf{j} &= -m(c)\nabla\mu \end{aligned}$$

leads the model of Lowengrub and Truskinovsky [58]

$$\rho D_t c + \operatorname{div}(m(c)\nabla\mu) = 0 \tag{3.28}$$

$$-\operatorname{div}(\rho f_{,\nabla c}) + \rho f_{,c} + \frac{1}{\rho}\hat{\rho}'(c)p = \rho\mu \tag{3.29}$$

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{v}) = 0 \tag{3.30}$$

$$\rho D_t\mathbf{v} - \operatorname{div}(2\eta(c)D\mathbf{v}) - \nabla(\lambda(c)\operatorname{div}\mathbf{v}) + \nabla p = -\nabla\cdot(\rho\nabla c\otimes f_{,\nabla c}). \tag{3.31}$$

One can now use $D_t\rho = \hat{\rho}'(c)D_t c$ to rewrite (3.30) as

$$\operatorname{div}\mathbf{v} = -(\hat{\rho})^{-2}\hat{\rho}'\operatorname{div}(m(c)\nabla\mu)\,. \tag{3.32}$$

Choosing

$$\hat{f}(c,\nabla c) = \tilde{\sigma}(\varepsilon|\nabla c|^2 + \frac{1}{\varepsilon}\psi(c))$$

gives that (3.29) and (3.31) become

$$\mu = \frac{\tilde{\sigma}}{\varepsilon}\psi'(c) - \frac{p}{\rho^2}\hat{\rho}'(c) - \frac{\tilde{\sigma}\varepsilon}{\rho}\operatorname{div}(\rho\nabla c)$$

and

$$\rho D_t\mathbf{v} - \operatorname{div}(2\eta(c)D\mathbf{v}) - \nabla(\lambda(c)\operatorname{div}\mathbf{v}) + \nabla p = -\tilde{\sigma}\varepsilon\operatorname{div}(\rho\nabla c\otimes\nabla c)\,.$$

Sometimes also the following ansatz for $\hat{f}$ is chosen

$$\hat{f}(c,\nabla c) = \frac{\hat{\sigma}}{\hat{\rho}(c)}(\varepsilon|\nabla c|^2 + \frac{1}{\varepsilon}\psi(c))\,,$$

see, e.g., [1, 4]. In this case (3.29) and (3.31) are given as

$$\rho\mu = \frac{\hat{\sigma}}{\varepsilon}\psi'(c) - \hat{\sigma}\varepsilon\Delta c + \frac{1}{\rho}\hat{\rho}'(c)p \tag{3.33}$$

and

$$\rho D_t\mathbf{v} - \operatorname{div}(2\eta(c)D\mathbf{v}) - \nabla(\lambda(c)\operatorname{div}\mathbf{v}) + \nabla p = -\hat{\sigma}\varepsilon\operatorname{div}(\nabla c\otimes\nabla c)\,.$$

In the case of a simple mixture, see (3.25), we have

$$\hat{\rho}(c) = \frac{1}{\alpha+\beta c}$$

with

$$\beta = \frac{1}{2\tilde{\rho}_-} - \frac{1}{2\tilde{\rho}_+}\ ,\ \alpha = \frac{1}{2\tilde{\rho}_+} + \frac{1}{2\tilde{\rho}_-}\,.$$

We hence obtain

$$\hat{\rho}'(c) = -\frac{\beta}{(\alpha+\beta c)^2} = -\beta\hat{\rho}(c)^2\,.$$

this then implies that (3.32) has the following simple divergence structure

$$\operatorname{div}(\mathbf{v} - \beta m(c)\nabla\mu) = 0\,.$$

In addition, (3.33) becomes

$$\mu = \frac{\hat{\sigma}}{\varepsilon\rho}\psi'(c) - \frac{\hat{\sigma}\varepsilon}{\rho}\Delta c - \beta p\,.$$

The major differences between the model studied in Section 3.1 which was based on a volume averaged velocity and the model studied in this section is that the model which is based on a mass balanced velocity leads to a velocity which in general is not divergence free and to a pressure dependent chemical potential. Both facts make the analysis of this model much more involved, cf. Section 3.5 below. We point out that both models reduce to "Model H" in the case that the two mass densities ρ_- and ρ_+ are the same.

3.3 Analytic Result in the Case of Same Densities

In this subsection we will discuss the mathematical results concerning existence and uniqueness of weak and strong solutions and partly their qualitative behavior for large times in the case that $\rho(c) \equiv$ *const.* In this case (3.17)-(3.20) as well as (3.28)-(3.31) reduce to the system

$$\rho\partial_t\mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}(2\eta(c)D\mathbf{v}) + \nabla p = -\operatorname{div}(\nabla c\otimes\nabla c), \tag{3.34}$$
$$\operatorname{div}\mathbf{v} = 0, \tag{3.35}$$
$$\partial_t c + \mathbf{v}\cdot\nabla c = \operatorname{div}(m(c)\nabla\mu), \tag{3.36}$$
$$\mu = -\varepsilon\Delta c + \frac{1}{\varepsilon}\psi'(c), \tag{3.37}$$

where we have chosen $\hat{f}(c,\nabla c) = \varepsilon|\nabla c|^2 + \frac{1}{\varepsilon}\psi(c)$. The system is studied in $\Omega\times(0,T)$, $T\in(0,\infty]$, where $\Omega\subseteq\mathbb{R}^d$, $d=2,3$, is a suitable domain, e.g., a bounded sufficiently smooth domain. It has to be closed by suitable initial and boundary conditions. The standard choice, which was done for most mathematical results, consists of

$$\mathbf{v}|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{3.38}$$
$$\boldsymbol{\nu}_{\partial\Omega}\cdot\nabla c|_{\partial\Omega} = \boldsymbol{\nu}_{\partial\Omega}\cdot\nabla\mu|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{3.39}$$
$$(\mathbf{v},c)|_{t=0} = (\mathbf{v}_0,c_0) \tag{3.40}$$

for suitable initial values $(\mathbf{v}_0, c_0)$. For all results mentioned in the following it is assumed that $\eta\colon\mathbb{R}\to(0,\infty)$ is sufficiently smooth, strictly positive and bounded. For existence of weak solutions continuity of η is usually sufficient. But more smoothness is needed for higher regularity and uniqueness. In the following we will assume for simplicity that $\rho = 1$. However, the results will also be true for general positive constant ρ.

Before we discuss the analytic results let us note that every sufficiently smooth solution of (3.34)-(3.40) on a suitable domain Ω (e.g., bounded with Lipschitz boundary) satisfies

$$\begin{aligned}&\frac{d}{dt}E(c(t),\mathbf{v}(t))\\&=-\int_\Omega 2\eta(c(x,t))|D\mathbf{v}(x,t)|^2\,dx-\int_\Omega m(c)|\nabla\mu(x,t)|^2\,dx\end{aligned}\tag{3.41}$$

for all $t\in(0,T)$, where $E(c,\mathbf{v})=E_{free}(c)+E_{kin}(\mathbf{v})$ and

$$\begin{aligned}E_{free}(c(t))&=\frac{\varepsilon}{2}\int_\Omega|\nabla c(x,t)|^2\,dx+\int_\Omega\frac{\psi(c(x,t))}{\varepsilon}\,dx,\\E_{kin}(\mathbf{v}(t))&=\frac{1}{2}\int_\Omega\rho|\mathbf{v}(x,t)|^2\,dx.\end{aligned}\tag{3.42}$$

This is a consequence of the energy dissipation inequality (3.15) integrated with respect to $x\in\Omega$ together with the boundary conditions (3.38)-(3.39). Alternatively, it follows from testing (3.34) with $\mathbf{v}$, (3.36) with μ and (3.37) with $\partial_t c$ as well as integration by parts. Moreover, it is often useful to replace (3.34) by

$$\rho\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v}-\operatorname{div}(2\eta(c)D\mathbf{v})+\nabla g=\mu\nabla c\tag{3.43}$$

with the new pressure $g=p+\frac{\varepsilon}{2}|\nabla c|^2+\varepsilon^{-1}\psi(c)$, cf. (3.23).

The analytic results often differ by their assumptions on the mobility m and the (homogeneous) free energy density ψ. Therefore we give a brief overview of these assumptions now. It is always assumed that $m\colon\mathbb{R}\to[0,\infty)$ is sufficiently smooth and bounded. Most of the time it is assumed that the mobility coefficient m is *non-degenerate*, which means that m is strictly positive. In the case of a *degenerate mobility* it is assumed that $m(c)=0$ if and only if $c\in\{a,b\}$, where $a,b\in\mathbb{R}$ represent the pure phases, which are $a=-1,b=+1$ in our derivation. Moreover, a suitable behavior of $m(c)$ as $c\to\pm1$ is assumed in the degenerate case. A canonical example is $m(c)=m_0(1-c^2)$ with $m_0>0$. A mathematical advantage of the degenerate case is that it prevents the concentration c from leaving the physical interval $[-1,1]$. But in most cases one even assumes that m is a positive constant (e.g., $m\equiv1$). A standard choice for ψ is that $\psi\colon\mathbb{R}\to\mathbb{R}$ is a sufficiently smooth function satisfying suitable growth conditions for $c\to\pm\infty$. From the physical point of view it should be of *double well type*, which in particular means that $\psi(c)\geq0$ with equality if and only if $c\in\{\pm1\}$. A canonical example is

$$\psi(c)=(1-c^2)^2,\qquad c\in\mathbb{R}.$$

But choosing such a smooth free energy density ψ has the mathematical disadvantage that there is no mechanism known, which prevents c from leaving the physical reasonable interval $[-1,1]$ even if the initial value c_0 attains only values in $[-1,1]$. One possibility to ensure that c stays in $[-1,1]$ is to choose ψ as a *singular free energy* density, e.g., of the form

$$\psi(c)=\frac{\theta}{2}\left((1+c)\ln(1+c)+(1-c)\ln(1-c)\right)-\frac{\theta_c}{2}c^2 \tag{3.44}$$

if $c\in[-1,1]$ and $\psi(c)=+\infty$ else. Here $0<\theta<\theta_c$ and $0\ln 0:=0=\lim_{s\to 0+}s\ln s$. Essential properties of this choice of ψ are

$$\psi'(s)\to_{(-1,1)\ni s\to\pm 1}\pm\infty, \qquad \inf_{s\in(-1,1)}\psi''(s)\geq-\theta_c>-\infty.$$

Using these properties it is possible to prove existence of weak (or strong) solutions with $c(x,t)\in(-1,1)$ for almost every $x\in\Omega$, $t\in(0,T)$, in many situations if the mobility is non-degenerate. Instead of ψ more general free energy densities with the latter properties can be considered. More details will be given below.

Now we discuss the analytic results in the case of matched densities (i.e., $\rho\equiv const.$) in more detail. A first result on existence of strong solutions, in the case that $\Omega=\mathbb{R}^2$ and ψ is a suitably smooth double well potential, was obtained by Starovoitov [74].

More complete results were presented by Boyer [29] in the case of a shear flow in a periodic channel. More precisely, it is assumed that

$$\Omega=\{x=(x',x_d)\in\mathbb{R}^d : x_d\in(-1,1)\}, d=2,3,$$

with periodic boundary conditions with respect to $x'\in\mathbb{R}^{d-1}$ and $\mathbf{v}|_{x_n=\pm1}=\pm Ue_1$ with $U>0$. Moreover, either the mobility m is non-degenerate and ψ is a suitable smooth potential or m is degenerate and $\psi=\psi_1+\psi_2$, where $\psi_1\colon(-1,1)\to\mathbb{R}$ is convex such that $m\psi_1''$ has a continuous extension on $[-1,1]$ and $\psi_2\in C^2([-1,1])$. We note that these assumptions are satisfied for ψ as in (3.44) if $\psi_1(c)=\frac{\theta}{2}\left((1+c)\ln(1+c)+(1-c)\ln(1-c)\right)$ and $m(c)=1-c^2$. In the case of non-degenerate mobility the existence of global weak solutions, which are strong and unique if either $d=2$ or $d=3$ and $t\in(0,T_0)$ for a sufficiently small $T_0>0$, was shown in [29]. Furthermore, in the degenerate case the existence of weak solutions with $c(t,x)\in[-1,1]$ almost everywhere is proved. The system (3.34)-(3.37) was also briefly discussed by Liu and Shen [57].

In the case of a singular free energy density and for constant positive mobility existence of weak solutions, strong well-posedness and convergence for large times was proven in [5]. We will describe these results now in more detail.

Assumption 3.1. *Let $\Omega\subseteq\mathbb{R}^d$ be a bounded domain with C^3-boundary and let $\psi\in C([-1,1])\cap C^2((-1,1))$ such that*

$$\lim_{s\to\pm1}\psi'(s)=\pm\infty, \qquad \psi''(s)\geq-\alpha \quad \textit{for all } s\in(-1,1)$$

for some $\alpha \in \mathbb{R}$. Furthermore, we assume that $\eta \in C^2([a,b])$ is a positive function. Finally, we extend $\psi(s)$ by $+\infty$ if $s \notin [-1,1]$.

Definition 3.2. (Weak Solution)
Let $0 < T \leq \infty$. A triple $(\mathbf{v}, c, \mu)$ such that

$$\mathbf{v} \in BC_w([0,T]; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_0(\Omega)^d),$$
$$c \in BC_w([0,T]; H^1(\Omega)), \ \ \psi'(c) \in L^2_{\text{loc}}([0,T); L^2(\Omega)), \nabla\mu \in L^2(\Omega \times (0,T))^d$$

is called a weak solution of (3.34)-(3.40) on $(0,T)$ if

$$-\int_0^T \int_\Omega \mathbf{v} \cdot \partial_t \boldsymbol{\psi} \, dx \, dt - \int_\Omega \mathbf{v}_0 \cdot \boldsymbol{\psi}|_{t=0} \, dx + \int_0^T \int_\Omega ((\mathbf{v} \cdot \nabla)\mathbf{v}) \cdot \boldsymbol{\psi} \, dx \, dt + \int_0^T \int_\Omega 2\eta(c) D\mathbf{v} : D\boldsymbol{\psi} \, dx \, dt = \int_0^T \int_\Omega \mu \nabla c \cdot \boldsymbol{\psi} \, dx \, dt \tag{3.45}$$

for all $\boldsymbol{\psi} \in C^\infty_{(0)}([0,T) \times \Omega)^d$ with $\operatorname{div} \boldsymbol{\psi} = 0$,

$$-\int_0^T \int_\Omega c \partial_t \varphi \, dx \, dt - \int_\Omega c_0 \varphi|_{t=0} \, dx + \int_0^T \int_\Omega \mathbf{v} \cdot \nabla c \, \varphi \, dx \, dt \, , = -\int_0^T \int_\Omega m(c) \nabla\mu \cdot \nabla\varphi \, dx \, dt \tag{3.46}$$

$$\int_0^T \int_\Omega \mu\varphi \, dx \, dt = \int_0^T \int_\Omega \psi'(c)\varphi \, dx \, dt + \int_0^T \int_\Omega \nabla c \cdot \nabla\varphi \, dx \, dt \tag{3.47}$$

for all $\varphi \in C^\infty_{(0)}([0,T) \times \overline{\Omega})$, and if the (strong) energy inequality

$$E(\mathbf{v}(t), c(t)) + \int_{t_0}^t \int_\Omega \left(2\eta(c)|D\mathbf{v}|^2 + |\nabla\mu|^2\right) \, dx \, d\tau \leq E(\mathbf{v}(t_0), c(t_0)) \tag{3.48}$$

holds for almost all $0 \leq t_0 < T$ including $t_0 = 0$ and all $t \in [t_0, T)$.

Here $L^2_\sigma(\Omega) = \overline{\{\boldsymbol{\varphi} \in C^\infty_0(\Omega)^d : \operatorname{div} \boldsymbol{\varphi} = 0\}}^{L^2(\Omega)}$, $BC_w([0,T]; X)$ is the space of all weakly continuous and bounded functions $f\colon [0,T] \to X$ and $L^2_{\text{loc}}([0,\infty); X)$ the space of all strongly measurable $f\colon [0,\infty) \to X$ such that $f|_{[0,T]} \in L^2(0,T;X)$ for all $T < \infty$, where X is a Banach space. Furthermore, in the following $BUC(I;X)$ denotes the space of all bounded and uniformly continuous $f\colon I \to X$ if $I \subseteq \mathbb{R}$ is an interval.

We note due to (3.41) sufficiently smooth solutions satisfy (3.48) with equality for all $0 \leq t_0 \leq t < T$. Moreover, this estimate motivates

$$\mathbf{v} \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_0(\Omega)^d),$$
$$c \in L^\infty(0,T; H^1(\Omega)), \quad \nabla\mu \in L^2(\Omega \times (0,T))^d.$$

As usual weak solutions are constructed by solving a suitable system, which approximates (3.34)-(3.40) and satisfies the same kind of energy inequality. Then one passes to the limit using the bounds in the spaces above. To this end one of the crucial points is to obtain a suitable bound on $\psi'(c)$. To this end the assumptions on ψ due to Assumption 3.1 play an essential role. If one defines $\psi_0(s) = \psi(s) + \alpha\frac{s^2}{2}$, then $\psi_0 \in C([-1,1]) \cap C^2((-1,1))$ is convex and satisfies

$$\psi_0(s) \to_{s\to\pm1} \pm\infty.$$

If one replaces ψ by ψ_0 in E_{free}, one obtains a lower semi-continuous, convex functional on

$$L^2_{(m)}(\Omega) := \{f \in L^2(\Omega) : \frac{1}{|\Omega|}\int_\Omega f(x)\,dx = m\},$$

with $m := \frac{1}{|\Omega|}\int_\Omega c_0(x)\,dx$. Its subgradient plays an important role in the analysis of (3.36)-(3.37) and can be characterized as follows:

Theorem 3.3. *Let ψ_0 be as above. Moreover, we set $\phi_0(x) = +\infty$ for $x \notin [-1,1]$ and let $E_0\colon L^2_{(m)}(\Omega) \to (-\infty,+\infty]$ be defined as*

$$E_0(c) := \int_\Omega \left(\frac{|\nabla c|^2}{2} + \psi_0(c)\right) dx$$

if $c \in H^1(\Omega)$ with $c(x) \in [-1,1]$ almost everywhere and $E_0(c) = +\infty$ else. Moreover, let ∂E_0 be its subgradient with respect to the L^2-inner product. Then

$$\begin{aligned}\mathcal{D}(\partial E_0) \quad = \quad & \Big\{c \in H^2(\Omega) \cap L^2_{(m)}(\Omega) : \\ & \quad \psi_0'(c) \in L^2(\Omega), \psi_0''(c)|\nabla c|^2 \in L^1(\Omega), \nu_{\partial\Omega}\cdot\nabla c|_{\partial\Omega} = 0\Big\}\end{aligned}$$

and

$$\partial E_0(c) = -\Delta c + P_0\psi_0'(c), \tag{3.49}$$

where $P_0\colon L^2(\Omega) \to L^2_{(0)}(\Omega)$ is the orthonormal projection onto $L^2_{(0)}(\Omega)$. Moreover, there is some $C > 0$ independent of $c \in \mathcal{D}(\partial E_0)$ such that

$$\begin{aligned}&\|c\|^2_{H^2(\Omega)} + \|\psi_0'(c)\|^2_{L^2(\Omega)} \\ &+ \int_\Omega \psi_0''(c(x))|\nabla c(x)|^2\,dx \le C\left(\|\partial E_0(c)\|^2_{L^2(\Omega)} + \|c\|^2_{L^2(\Omega)} + 1\right).\end{aligned} \tag{3.50}$$

The result was proven by Abels and Wilke [18, Theorem 4.3]. Formally, one can obtain (3.50) by multiplying $\partial E_0(c) = -\Delta c + P_0\psi_0'(c)$ by $-\Delta c$. This yields

$$\begin{aligned}-\int_\Omega \partial E_0(c)\Delta c\,dx &= \int_\Omega |\Delta c|^2\,dx - \int_\Omega P_0(\psi_0'(c))\Delta c\,dx \\ &= \int_\Omega |\Delta c|^2\,dx + -\int_\Omega \underbrace{\nabla\psi_0'(c)\cdot\nabla c}_{=\psi_0''(c)|\nabla c|^2 \ge 0}\,dx \ge \|\Delta c\|^2_{L^2(\Omega)}.\end{aligned}$$

Using regularity results for elliptic equations and Young's inequality one obtains (3.50) formally. These formal arguments are justified rigorously in the proof of [18, Theorem 4.3]. Using (3.50) together with the a priori estimates for c and μ from the energy inequality, one obtains a bound on $\nabla^2 c, \psi'(c) \in L^2_{\text{loc}}([0,\infty); L^2(\Omega))$. Based on this one obtains:

Theorem 3.4. (Existence of Weak Solutions, [5, Theorem 1])
Let $m > 0$ be independent of c. Then for every $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $c_0 \in H^1(\Omega)$ with $c_0(x) \in [-1,1]$ almost everywhere there is a weak solution $(\mathbf{v}, c, \mu)$ of (3.34)-(3.40) on $(0,\infty)$. Moreover, if $d = 2$, then (3.48) holds with equality for all $0 \le t_0 \le t < \infty$. Finally, every weak solution on $(0,\infty)$ satisfies

$$\nabla^2 c, \psi'(c) \in L^2_{loc}([0,\infty); L^r(\Omega)), \frac{t^{\frac{1}{2}}}{1+t^{\frac{1}{2}}} c \in BUC([0,\infty); W^1_q(\Omega)) \quad (3.51)$$

where $r = 6$ if $d = 3$ and $1 < r < \infty$ is arbitrary if $d = 2$ and $q > 3$ is independent of the solution and initial data. If additionally $c_0 \in H^2_N(\Omega) := \{u \in H^2(\Omega) : \nu_{\partial\Omega} \cdot \nabla u|_{\partial\Omega} = 0\}$ and $-\Delta c_0 + \psi_0'(c_0) \in H^1(\Omega)$, then we have $c \in BUC([0,\infty); W^1_q(\Omega))$.

We note that $\nabla^2 c, \psi'(c) \in L^2_{loc}([0,\infty); L^r(\Omega))$ in (3.51) follows from a generalization of (3.50), Theorem 3.3, resp., for $L^r(\Omega)$ instead of $L^2(\Omega)$, cf. [5, Lemma 2]. For further regularity studies and uniqueness results, it is important that Theorem 3.4 provides $c \in BUC(\delta, \infty; W^1_q(\Omega))$ for some $q > d$ and for all $\delta > 0$ and $\delta = 0$ for suitable initial data. This makes it possible to use a result on maximal regularity for an associated Stokes system with variable viscosity, cf. [5, Proposition 4], to conclude higher regularity for the velocity $\mathbf{v}$ in the case of small or large times and in the case $d = 2$, which is enough to obtain a (locally) unique solution. Then one obtains:

Theorem 3.5. (Uniqueness, [5, Proposition 1])
Let $m > 0$ be independent of c, $0 < T \le \infty$, $q = 3$ if $d = 3$ and let $q > 2$ if $d = 2$. Moreover, assume that $\mathbf{v}_0 \in W^1_{q,0}(\Omega) \cap L^2_\sigma(\Omega)$ and let $c_0 \in H^1(\Omega) \cap C^{0,1}(\overline{\Omega})$ with $c_0(x) \in [-1,1]$ for all $x \in \Omega$. If there is a weak solution $(\mathbf{v}, c, \mu)$ of (3.34)-(3.40) on $(0,T)$ with $\mathbf{v} \in L^\infty(0,T; W^1_q(\Omega))$ and $\nabla c \in L^\infty(\Omega \times (0,T))$, then any weak solution $(\mathbf{v}', c', \mu')$ of (3.34)-(3.40) on $(0,T)$ with the same initial values and $\nabla c' \in L^\infty(\Omega \times (0,T))^d$ coincides with $(\mathbf{v}, c, \mu)$.

For the following we denote $V_2^{1+j}(\Omega) = H^{1+j}(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$, $j = 0, 1$. Moreover, for $s \in (0,1)$ we define $V_2^{1+s}(\Omega) = (V_2^1(\Omega), V_2^2(\Omega))_{s,2}$, where $(.,.)_{s,q}$ denotes the real interpolation functor.

Theorem 3.6. (Regularity of Weak Solutions, [5, Theorem 2])
Let $m > 0$ be independent of c and let $c_0 \in H^2_N(\Omega)$ such that $E_{free}(c_0) < \infty$ and $-\Delta c_0 + \psi'(c_0) \in H^1(\Omega)$.

(*i*) *Let* $d = 2$ *and let* $\mathbf{v}_0 \in V_2^{1+s}(\Omega)$ *with* $s \in (0,1]$. *Then every weak solution* $(\mathbf{v}, c)$ *of (3.34)-(3.40) on* $(0,\infty)$ *satisfies*

$$\mathbf{v} \in L^2(0,\infty; H^{2+s'}(\Omega)) \cap H^1(0,\infty; H^{s'}(\Omega)) \cap BUC([0,\infty); H^{1+s-\varepsilon}(\Omega))$$

for all $s' \in [0,\frac{1}{2}) \cap [0,s]$ *and all* $\varepsilon > 0$ *as well as* $\nabla^2 c, \psi'(c) \in L^\infty(0,\infty; L^r(\Omega))$ *for every* $1 < r < \infty$. *In particular, the weak solution is unique.*

(*ii*) *Let* $d = 2, 3$. *Then for every weak solution* $(\mathbf{v}, c, \mu)$ *of (3.34)-(3.40) on* $(0,\infty)$ *there is some* $T > 0$ *such that*

$$\mathbf{v} \in L^2(T,\infty; H^{2+s}(\Omega)) \cap H^1(T,\infty; H^{s}(\Omega)) \cap BUC([T,\infty); H^{2-\varepsilon}(\Omega))$$

for all $s \in [0,\frac{1}{2})$ *and all* $\varepsilon > 0$ *as well as* $\nabla^2 c, \psi'(c) \in L^\infty(T,\infty; L^r(\Omega))$ *with* $r = 6$ *if* $d = 3$ *and* $1 < r < \infty$ *if* $d = 2$.

(*iii*) *If* $d = 3$ *and* $\mathbf{v}_0 \in V_2^{s+1}(\Omega)$, $s \in (\frac{1}{2}, 1]$, *then there is some* $T_0 > 0$ *such that every weak solution* $(\mathbf{v}, c)$ *of (3.34)-(3.40) on* $(0, T_0)$ *satisfies*

$$\mathbf{v} \in L^2(0,T_0; H^{2+s'}(\Omega)) \cap H^1(0,T_0; H^{s'}(\Omega)) \cap BUC([0,T_0]; H^{1+s-\varepsilon}(\Omega))$$

for all $s' \in [0,\frac{1}{2})$ *and all* $\varepsilon > 0$ *as well as* $\nabla^2 c, \psi'(c) \in L^\infty(0,T_0; L^6(\Omega))$. *In particular, the weak solution is unique on* $(0, T_0)$.

The proof of the latter theorem is essentially based on the fact that $c \in BUC([0,\infty); W_q^1(\Omega))$ for some $q > d$, which implies that $c \colon \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is uniformly continuous. This makes it possible to use regularity results for the Stokes system with variable viscosity $\eta(c)$, which is the linearization of the right-hand side of (3.34), together with regularity results for the Cahn-Hilliard equation with convection term (3.36)-(3.37).

We note that similar results on existence of weak solutions can also be obtained for the so-called double obstacle potential for ψ, i.e.,

$$\psi(c) = \begin{cases} -\frac{\theta_c}{2} c^2 & \text{if } c \in [-1,1], \\ +\infty & \text{else.} \end{cases}$$

But in this case (3.37) has to be replaced by the differential inclusion

$$\mu + \Delta c + \theta_c c \in \partial I_{[-1,1]}(c),$$

where $I_{[-1,1]}$ is the indicator function of $[-1,1]$, i.e., $I_{[-1,1]}(c) = 0$ if $c \in [-1,1]$ and $I_{[-1,1]}(c) = +\infty$ else. This double obstacle potential is the pointwise limit of ψ in (3.44), when $\theta \to 0$, cf. Figure 3. It can also be shown that the corresponding solutions of (3.34)-(3.40) converge as $\theta \to 0$ to solutions of the system (3.34)-(3.40), cf. [1, Section 6.5] or [6].

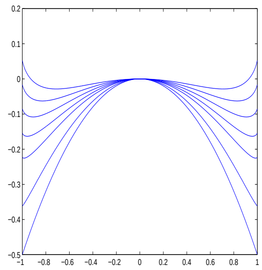

Figure 3: Logarithmic free energy in (3.44) for $\theta \to 0$.

Finally, we note that, because of the regularity of any weak solution for large times, it is possible to prove convergence to stationary solutions as $t \to \infty$.

Theorem 3.7. (Convergence to Stationary Solution)
Assume that $\psi\colon (-1,1) \to \mathbb{R}$ *is analytic and let* $(\mathbf{v}, c, \mu)$ *be a weak solution of (3.34)-(3.40). Then* $(\mathbf{v}(t), c(t)) \rightharpoonup_{t\to\infty} (0, c_\infty)$ *in* $H^{2-\varepsilon}(\Omega)^d \times H^2(\Omega)$ *for all* $\varepsilon > 0$ *and for some* $c_\infty \in H^2(\Omega)$ *with* $\psi'(c_\infty) \in L^2(\Omega)$ *solving the stationary Cahn-Hilliard equation*

$$-\Delta c_\infty + \psi'(c_\infty) = const. \qquad \text{in } \Omega, \tag{3.52}$$

$$\nu_{\partial\Omega} \cdot \nabla c_\infty|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega, \tag{3.53}$$

$$\int_\Omega c_\infty(x)\,dx = \int_\Omega c_0(x)\,dx. \tag{3.54}$$

The proof is based on the so-called Lojasiewicz-Simon inequality, cf. [5] for details. To prove this inequality it is important that $\psi\colon (-1,1) \to \mathbb{R}$ is analytic, which is the case for the canonical example (3.44).

Finally, we note that (3.34)-(3.37) was also considered in the case of non-Newtonian fluids of power-law type. In this case $2\eta(c)D\mathbf{v}$ in (3.34) is replaced by general viscous stress tensor $\mathbf{S}(c, D\mathbf{v})$, which satisfies suitable growth conditions with respect an exponent $p > 1$. First analytic results in this case were obtained by Kim, Consiglieri, and Rodrigues [52]. They proved existence of weak solutions in the case $p \geq \frac{3d+2}{d+2}$, $d = 2, 3$, using monotone operator techniques. In [48] Grasselli and Pražak discussed the longtime behavior of solutions of the system in the case $p \geq \frac{3d+2}{d+2}$, $d = 2, 3$, in the case of periodic boundary conditions and a regular free energy density. For the same p existence of weak solutions with a singular free energy density f was proved by Bosia [28] in the case of a bounded domain in $\mathbb{R}^3$. Moreover, the longtime behavior was studied. Finally, existence of weak solutions was shown by Abels, Diening, and Terasawa [11] in the case that $p > \frac{2d}{d+2}$ using the parabolic Lipschitz truncation method for divergence free vector fields developed by Breit, Diening, and Schwarzacher [32],

3.4 Analysis for the Model with General Densities Based on the Volume Averaged Velocity

In this subsection we discuss analytic results of the system (3.17)-(3.20), i.e.,

$$\rho\partial_t \mathbf{v} + ((\rho\mathbf{v} + \widetilde{\mathbf{J}})\cdot\nabla)\mathbf{v} - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \nabla p = -\varepsilon \operatorname{div}(\nabla\varphi\otimes\nabla\varphi), \quad (3.55)$$

$$\operatorname{div}\mathbf{v} = 0, \quad (3.56)$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \operatorname{div}(m(\varphi)\nabla\mu), \quad (3.57)$$

$$\mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}\psi'(\varphi) \quad (3.58)$$

in $\Omega\times(0,T)$, where $\Omega\subseteq\mathbb{R}^d$, $d=2,3$, is a bounded domain with smooth boundary and

$$\begin{aligned}\rho = \rho(\varphi) &= \tilde{\rho}_+\frac{1+\varphi}{2} + \tilde{\rho}_-\frac{1-\varphi}{2}, \\ \widetilde{\mathbf{J}} &= \frac{\tilde{\rho}_+-\tilde{\rho}_-}{2}\mathbf{J}_\varphi = -\frac{\tilde{\rho}_+-\tilde{\rho}_-}{2}m(\varphi)\nabla\mu.\end{aligned} \quad (3.59)$$

We close the system with the boundary and initial conditions (3.38)-(3.40). Here we have set $\hat{\sigma}=1$ for simplicity.

Smooth solutions of (3.55)-(3.58) together with (3.38)-(3.40) satisfy the same energy dissipation identity as in the case of same densities, i.e., (3.41), where c is replaced by φ and $\rho=\rho(\varphi)$ in (3.42). In particular, this yields a priori bounds for

$$\begin{aligned}&\mathbf{v}\in L^\infty 0,\infty; L^2_\sigma(\Omega))\cap L^2(0,\infty;H^1_0(\Omega)^d), \varphi\in L^\infty(0,\infty;H^1(\Omega)), \\ &\nabla\mu\in L^2(0,\infty;L^2(\Omega))^d \text{ if } m(\varphi)\geq m_0>0.\end{aligned}$$

as in the case of same densities.

So far there are only few results on existence of solutions to the system above. The system was discussed by Abels, Depner, and Garcke in [10] and [9], where existence of weak solutions in the case of singular free energies with non-degenerate and degenerate mobility, respectively, was shown. More precisely, in the non-degenerate case the following result was shown:

Theorem 3.8. (Existence of Weak Solutions, [10, Theorem 3.4]) *Let $m\in C^1(\mathbb{R})$ be bounded such that $\inf_{s\in\mathbb{R}} m(s)>0$, let Assumption 3.1 hold true and assume that additionally $\lim_{s\to\pm1}\frac{\psi''(s)}{\psi'(s)}=+\infty$. Then for every $\mathbf{v}_0\in L^2_\sigma(\Omega)$ and $\varphi_0\in H^1(\Omega)$ with $|\varphi_0|\leq 1$ almost everywhere and $\frac{1}{|\Omega|}\int_\Omega\varphi_0\,dx\in(-1,1)$ there exists a weak solution $(\mathbf{v},\varphi,\mu)$ of* (3.55)-(3.58) *together with* (3.38)-(3.40) *such that*

$$\begin{aligned}&\mathbf{v}\in BC_w([0,\infty);L^2_\sigma(\Omega))\cap L^2(0,\infty;H^1_0(\Omega)^d), \\ &\varphi\in BC_w([0,\infty);H^1(\Omega))\cap L^2_{loc}([0,\infty);H^2(\Omega)),\ \psi'(\varphi)\in L^2_{loc}([0,\infty);L^2(\Omega)), \\ &\mu\in L^2_{loc}([0,\infty);H^1(\Omega)) \ \textit{with}\ \nabla\mu\in L^2(0,\infty;L^2(\Omega))^d.\end{aligned}$$

Here the definition of weak solutions is similar to Definition 3.2. We refer to [10, Definition 3.3] for the details.

The structure of the proof of Theorem 3.8 is as follows: System (3.55)-(3.58) is first approximated with the aid of a semi-implicit time discretization, which satisfies the same kind of energy identity as the continuous system. Hence one obtains a priori bounds for

$$\begin{aligned}
&\mathbf{v}^N \in L^\infty 0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; H^1_0(\Omega)^d)\,, \varphi^N \in L^\infty(0,\infty; H^1(\Omega))\,,\\
&\nabla\mu^N \in L^2(0,\infty; L^2(\Omega)^d) \text{ if } m(\varphi) \geq m_0 > 0,
\end{aligned}$$

where $(\mathbf{v}^N, \varphi^N, \mu^N)$ are suitable interpolations of the time discretized system with discretization parameter $h = \frac{1}{N}$. In order pass to the limit $N \to \infty$ it is essential to obtain a bound for

$$\varphi^N \in L^2_{loc}([0,\infty), H^2(\Omega))\,, \quad \psi'(\varphi^N) \in L^2_{loc}([0,\infty), L^2(\Omega)),$$

which follows from Theorem 3.3. The latter theorem is also used to obtain existence of solutions for the time discrete system with the aid of the Leray-Schauder principle and the theory of monotone operators.

In the case of degenerate mobility it is assumed that $\Psi \in C^1(\mathbb{R})$,

$$m(s) = \begin{cases} 1 - s^2 & \text{if } s \in [-1,1], \\ 0 & \text{else} \end{cases}$$

and η and Ω are as in Assumption 3.1. In this case one does not obtain an a priori bound for $\nabla\mu$ in $L^2((0,T)\times\Omega)$. Instead one obtains an a priori bound for $\widehat{\mathbf{J}} := \sqrt{m(\varphi)}\nabla\mu$ and $\mathbf{J} := m(\varphi)\nabla\mu$. There one has to avoid $\nabla\mu$ in the weak formulation and has to formulate the equations in terms of $\mathbf{J}$. More precisely, weak solutions are defined as follows, cf. [9, Definition 3.3].

Definition 3.9. *Let $T \in (0,\infty)$, $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ and $\varphi_0 \in H^1(\Omega)$ with $|\varphi_0| \leq 1$ almost everywhere in Ω. Then we call the triple $(\mathbf{v}, \varphi, \mathbf{J})$ with the properties*

$$\begin{aligned}
&\mathbf{v} \in BC_w([0,T]; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_0(\Omega)^d)\,,\\
&\varphi \in BC_w([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \ \textit{ with } \ |\varphi| \leq 1 \ \textit{ a.e. in } Q_T\,,\\
&\mathbf{J} \in L^2(0,T; L^2(\Omega)^d) \ \textit{ and}\\
&(\mathbf{v}, \varphi)|_{t=0} = (\mathbf{v}_0, \varphi_0)
\end{aligned}$$

a weak solution of (3.55)-(3.58) *together with* (3.38)-(3.40) *if the following conditions are satisfied:*

$$\begin{aligned}
&-\int_0^T \int_\Omega \rho\mathbf{v} \cdot \partial_t\boldsymbol{\psi}\, dx\, dt + \int_0^T \int_\Omega \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) \cdot \boldsymbol{\psi}\, dx\, dt\\
&+ \int_0^T \int_\Omega 2\eta(\varphi) D\mathbf{v} : D\boldsymbol{\psi}\, dx\, dt - \int_0^T \int_\Omega (\mathbf{v} \otimes \tfrac{\tilde\rho_2 - \tilde\rho_1}{2}\mathbf{J}) : \nabla\boldsymbol{\psi}\, dx\, dt \qquad (3.60)\\
&= -\int_0^T \int_\Omega \Delta\varphi\, \nabla\varphi \cdot \boldsymbol{\psi}\, dx\, dt
\end{aligned}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega \times (0,T))^d$ *with* $\operatorname{div} \boldsymbol{\psi} = 0$,

$$-\int_0^T \int_\Omega \varphi\, \partial_t \zeta \, dx\, dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \varphi)\, \zeta \, dx\, dt = \int_0^T \int_\Omega \mathbf{J} \cdot \nabla \zeta \, dx\, dt \quad (3.61)$$

for all $\zeta \in C_0^\infty((0,T; C^1(\overline{\Omega}))$ *and*

$$\begin{aligned}\int_0^T \int_\Omega \mathbf{J} \cdot \boldsymbol{\eta}\, dx\, dt & \qquad (3.62)\\ = -\int_0^T \int_\Omega \left(\sqrt{a(\varphi)} \left(\widetilde{\Psi}'(A(\varphi)) - \Delta A(\varphi) \right) \right) \operatorname{div}(m(\varphi)\boldsymbol{\eta})\, dx\, dt &\end{aligned}$$

for all $\boldsymbol{\eta} \in L^2(0,T; H^1(\Omega)^d) \cap L^\infty(\Omega \times (0,T))^d$ *which fulfill* $\nu_{\partial\Omega} \cdot \boldsymbol{\eta}|_{\partial\Omega} = 0$ *on* $\partial\Omega \times (0,T)$.

We note that (3.62) is a weak formulation of

$$\mathbf{J} = -m(\varphi)\, \nabla \left(\sqrt{a(\varphi)} \left(\widetilde{\Psi}'(A(\varphi)) - \Delta A(\varphi) \right) \right) .$$

Theorem 3.10. (Existence of Weak Solutions, [9, Theorem 3.5]) *Let the previous assumptions hold,* $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ *and* $\varphi_0 \in H^1(\Omega)$ *with* $|\varphi_0| \le 1$ *almost everywhere in* Ω. *Then there exists a weak solution* $(\mathbf{v}, \varphi, \mathbf{J})$ *of* (3.55)-(3.58) *together with* (3.38)-(3.40) *in the sense of Definition 3.9. Moreover for some* $\widehat{\mathbf{J}} \in L^2(\Omega \times (0,T))$ *it holds that* $\mathbf{J} = \sqrt{m(\varphi)}\widehat{\mathbf{J}}$ *and*

$$\begin{aligned} E(\varphi(t), \mathbf{v}(t)) + \int_s^t \int_\Omega 2\eta(\varphi)\, |D\mathbf{v}|^2 \, dx\, d\tau + \int_s^t \int_\Omega |\widehat{\mathbf{J}}|^2 \, dx\, d\tau \\ \le E_{tot}(\varphi(s), \mathbf{v}(s)) \end{aligned} \quad (3.63)$$

for all $t \in [s,T)$ *and almost all* $s \in [0,T)$ *including* $s = 0$, *where* $E(\varphi(t), \mathbf{v}(t))$ *is defined as in* (3.41) *with* $c(t)$ *replaced by* $\varphi(t)$. *In particular,* $\mathbf{J} = 0$ *a.e. on the set* $\{|\varphi| = 1\}$.

The theorem is proved by approximating m by a sequence of strictly positive mobilities m_ε and ψ by

$$\psi_\varepsilon(s) := \psi(s) + \varepsilon(1+s)\ln(1+s) + \varepsilon(1-s)\ln(1-s), \quad s \in [-1,1],$$

where $\varepsilon > 0$. Then existence of weak solutions $(\mathbf{v}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ for $\varepsilon > 0$ follows from Theorem 3.8. In order to pass to the limit one uses the energy inequality (3.48). But this does not give a bound for $\varphi_\varepsilon \in L^2(0,T; H^2(\Omega))$, which is essential to pass to the limit in the weak formulation of (3.55). In order to obtain this bound one tests the weak formulation of (3.57) with $G'_\varepsilon(\varphi_\varepsilon)$, where $G''(s) = \frac{1}{m_\varepsilon(s)}$ for $s \in (-1,1)$ and $G'_\varepsilon(0) = G_\varepsilon(0) = 0$. We refer to [9, Proof of Lemma 3.7] for the details.

Finally, we note that existence of weak solutions of (3.55)-(3.58) together with (3.34)-(3.37) was proven in the case of power-law type fluids of exponent $p > \frac{2d+2}{d+2}$, $d = 2, 3$, in [8]. More precise, $2\eta(\varphi)D\mathbf{v}$ in (3.55) is replaced by $\mathbf{S}(\varphi, D\mathbf{v})$, where $\mathbf{S}\colon \mathbb{R} \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ satisfies

$$\begin{aligned} |\mathbf{S}(s,\mathbf{M})| &\leq C(|\mathbf{M}|^{p-1}+1)\,, \\ |\mathbf{S}(s_1,\mathbf{M})-\mathbf{S}(s_2,\mathbf{M})| &\leq C|s_1-s_2|(|\mathbf{M}|^{p-1}+1)\,, \\ \mathbf{S}(s,\mathbf{M}):\mathbf{M} &\geq \omega|\mathbf{M}|^p - C_1 \end{aligned}$$

for all $\mathbf{M} \in \mathbb{R}^{d\times d}_{sym}$, $s, s_1, s_2 \in \mathbb{R}$, and some $C, C_1, \omega > 0$. Furthermore, the case of constant, positive mobility together with a suitable smooth free energy density ψ is considered. Unfortunately, in this case there is no mechanism, which enables to show that $\psi \in [-1, 1]$. Hence one has to modify ρ, defined as in (3.59) for $\varphi \in [-1, 1]$, outside of $[-1, 1]$ suitably such that it stays positive. But then (3.13) is no longer valid and one obtains instead

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{v} + \widehat{\mathbf{J}}) = R, \quad \text{where } R = -\nabla\frac{\partial\rho}{\partial\varphi}\cdot\nabla\mu. \tag{3.64}$$

Here R is an additional source term, which vanishes in the interior of $\{\varphi \in [-1, 1]\}$. In order to obtain a local dissipation inequality and global energy estimate the equation of linear momentum (3.55) has to be modified to

$$\varrho\partial_t\mathbf{v} + (\varrho\mathbf{v} + \widehat{\mathbf{J}}))\cdot\nabla\mathbf{v} + R\frac{\mathbf{v}}{2} - \operatorname{div}\mathbf{S}(\varphi, D\mathbf{v}) + \nabla p = -\varepsilon\operatorname{div}\left(\nabla\varphi\otimes\nabla\varphi\right).$$

Under these assumptions existence of weak solutions is shown with the aid of the so-called L^∞-truncation method, cf. [8] for the details.

3.5 Analysis for the Model with General Densities based on the Mass Averaged Velocity

In the following we discuss the known result on existence of weak and strong solutions for the model by Lowengrub and Tuskinovsky [58], i.e.,

$$\rho\partial_t\mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{S}(c, D\mathbf{v}) + \nabla p = -\varepsilon\operatorname{div}(|\nabla c|^{q-2}\nabla c\otimes\nabla c), \tag{3.65}$$

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{v}) = 0, \tag{3.66}$$

$$\rho\partial_t c + \rho\mathbf{v}\cdot\nabla c = \operatorname{div}(m(c)\nabla\mu), \tag{3.67}$$

$$\rho\mu = -\rho^{-1}\frac{\partial\rho}{\partial c}p - \operatorname{div}(\rho(c)|\nabla c|^{q-2}\nabla c) + \rho\psi'(c), \tag{3.68}$$

cf. (3.28)-(3.31), in $\Omega\times(0,T)$, where $\rho = \hat\rho(c)$ with

$$\frac{1}{\rho(c)} = \frac{1}{\tilde\rho_1}\frac{1-c}{2} + \frac{1}{\tilde\rho_2}\frac{1+c}{2}, \quad \mathbf{S}(c, D\mathbf{v}) = 2\eta(c)D\mathbf{v} + \lambda(c)\operatorname{div}\mathbf{v}\,\mathrm{Id},$$

$\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with C^3-boundary and $T \in (0, \infty]$. Moreover, we have chosen

$$\hat{f}(c, \nabla c) = \varepsilon^{q-1} \frac{|\nabla c|^q}{q} + \frac{\psi(c)}{\varepsilon}$$

for some $q \geq 2$. Usually one chooses $q = 2$ for these kinds of diffuse interface models. But for proving existence of weak solutions it is necessary so far to choose $q > d$. The reasons will be explained below.

We close the system by adding the boundary and initial conditions

$$\nu_{\partial\Omega} \cdot \mathbf{v}|_{\partial\Omega} = \nu_{\partial\Omega} \cdot \mathbf{S}(c, D\mathbf{v})_\tau + \gamma \mathbf{v}_\tau|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{3.69}$$

$$\nu_{\partial\Omega} \cdot \nabla c|_{\partial\Omega} = \nu_{\partial\Omega} \cdot \nabla \mu|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{3.70}$$

$$(\mathbf{v}, c)|_{t=0} = (\mathbf{v}_0, c_0) \quad \text{in } \Omega, \tag{3.71}$$

where $0 < \gamma < \infty$ is a friction coefficient and τ denotes the tangential part of a vector field. For the analysis it is important that we use Navier boundary conditions for $\mathbf{v}$ (3.69) instead of no-slip boundary conditions $\mathbf{v}|_{\partial\Omega} = 0$ as before since this makes it possible to estimate the pressure suitably.

In the following it is assumed that $\tilde{\rho}_1 \neq \tilde{\rho}_2$, that $\eta, m, \lambda, \psi \colon \mathbb{R} \to \mathbb{R}$ are sufficiently smooth and that η, λ, m are strictly positive and bounded. We refer to [4, 7] for the precise assumptions. Similar as for the other models smooth solutions of (3.65)-(3.71) satisfy the energy dissipation identity

$$\begin{aligned} &\frac{d}{dt} E(c(t), \mathbf{v}(t)) \\ &= -\int_\Omega \left(2\eta(c)|D\mathbf{v}|^2 + \lambda(c)|\operatorname{div} \mathbf{v}|^2\right) dx - \int_\Omega m(c)|\nabla \mu|^2 \, dx \end{aligned} \tag{3.72}$$

for all $t \in (0, T)$, where $E(c, \mathbf{v}) = E_{free}(c) + E_{kin}(\mathbf{v})$ and

$$\begin{aligned} E_{free}(c(t)) &= \int_\Omega \varepsilon^{q-1} \frac{|\nabla c(x,t)|^q}{q} \, dx + \int_\Omega \frac{\psi(c(x,t))}{\varepsilon} \, dx, \\ E_{kin}(\mathbf{v}(t)) &= \int_\Omega \rho(c(x,t)) \frac{|\mathbf{v}(x,t)|^2}{2} \, dx. \end{aligned}$$

In order to get a priori estimates for the construction of weak solutions it is essential that $\rho = \hat{\rho}(c)$ stays positive. We note that

$$\hat{\rho}(c) = \frac{1}{\alpha + \beta c}, \quad \text{where } \beta = \frac{1}{2\bar{\rho}_1} - \frac{1}{2\bar{\rho}_2}, \alpha = \frac{1}{2\bar{\rho}_2} + \frac{1}{2\bar{\rho}_1}$$

and

$$\hat{\rho}'(c) = -\beta^2 \hat{\rho}(c)^2,$$

as seen in Section 3.2. Hence we need a mechanism, which guarantees that c stays in $[-1, 1]$ or at least in $[-1 - \delta, 1 + \delta]$ for some sufficiently small

$\delta > 0$. Unfortunately, so far it was not possible to work with a singular free energy because of the pressure appearing in (3.67). But an alternative is to choose $q > d$, which yields an a priori bound for $c \in L^\infty(0,T;W^1_q(\Omega)) \hookrightarrow L^\infty(0,T;C^{1-\frac{d}{q}}(\overline{\Omega}))$. In this case c can be trapped in $[-1-\delta, 1+\delta]$ if ψ is chosen "steep enough" outside of the physical interval $[-1,1]$. More precisely we have:

Lemma 3.11. ([4, Lemma 2.3]) *Let $R, \delta > 0$, $q > d$, and let $\psi \in C^2([-1,1])$ with $\psi(c) > 0$, $c \in [-1,1]$, be given. Then there is an extension $\psi \in C^2(\mathbb{R})$, $\psi(c) \geq 0, \psi''(c) \geq -M > -\infty$ such that for all $c \in W^1_q(\Omega)$*

$$\int_\Omega \frac{\hat{\rho}(c)|\nabla c|^q}{q}\,dx + \int_\Omega \hat{\rho}(c)\psi(c)\,dx \leq R \quad \Rightarrow \quad c(x) \in (-1-\delta, 1+\delta). \tag{3.73}$$

In order to construct weak or strong solutions it is essential to reformulate (3.65)-(3.68) first. To this end we define

$$g = \psi(c) + \frac{|\nabla c|^q}{q} + \frac{p}{\rho} - \bar{\mu}c,$$

and $\mu = \mu_0 + \bar{\mu}$, $\bar{\mu} = \frac{1}{|\Omega|}\int_\Omega \mu\,dx$. Moreover, we decompose $g = g_0 + \bar{g}$, $\bar{g} = \frac{1}{|\Omega|}\int_\Omega g\,dx$. Then (3.65)-(3.68) are equivalent to

$$\rho\partial_t \mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{S}(c,\mathbf{D}\mathbf{v}) + \rho\nabla g_0 = \rho\mu_0\nabla c, \tag{3.74}$$

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{v}) = 0, \tag{3.75}$$

$$\rho\partial_t c + \rho\mathbf{v}\cdot\nabla c = \operatorname{div}(m(c)\nabla\mu_0), \tag{3.76}$$

$$\rho\mu_0 + \rho^2\bar{g} = \beta\rho^2 g_0 - \operatorname{div}(\rho(c)|\nabla c|^{q-2}\nabla c) + \psi'(c) \tag{3.77}$$

together with

$$\int_\Omega \mu_0(t)\,dx = \int_\Omega g_0(t)\,dx = 0 \quad \text{for all } t \in (0,T), \tag{3.78}$$

cf. [4, Section 3] for the details. Here the specific form of $\hat{\rho}$ and the corresponding relations above are essentially used.

For the mathematical analysis it is essential to use a suitable decomposition of g_0, namely:

$$g_0 = g_1 - \partial_t G(\mathbf{v}), \tag{3.79}$$

where

$$\Delta G(\mathbf{v}) = \operatorname{div}\mathbf{v} \qquad \text{in } \Omega, \tag{3.80}$$

$$\nu_{\partial\Omega}\cdot\nabla G(\mathbf{v})|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega, \tag{3.81}$$

and $\int_\Omega G(\mathbf{v})\,dx = 0$. This implies

$$\nabla G(\mathbf{v}) = (I - P_\sigma)\mathbf{v}, \tag{3.82}$$

where $P_\sigma\colon L^2(\Omega)^d \to L^2_\sigma(\Omega)$ is the Helmholtz projection. Hence (3.74) is equivalent to

$$\rho\partial_t P_\sigma \mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{S}(c, D\mathbf{v}) + \rho\nabla g_1 = \rho\mu_0\nabla c. \tag{3.83}$$

Here the part g_1 has relatively good regularity, e.g., $g_1 \in L^2(0,\infty; L^p(\Omega))$ with $1 < p < \frac{d}{d-1}$, cf. Theorem 3.13 below. It is the part $\partial_t G(\mathbf{v})$, which makes the analysis difficult and which does not allow to use a singular free energy as in (3.44). – We also note that for the estimates of g_1 it is important to consider Navier boundary conditions for $\mathbf{v}$ and not no-slip boundary conditions.

Because of (3.83) one defines:

Definition 3.12. *Let $\mathbf{v}_0 \in L^2(\Omega)^d$, $c_0 \in W^1_q(\Omega)$, $q > d$, and let $\psi\colon \mathbb{R} \to [0,\infty)$ be twice continuously differentiable. Then $(\mathbf{v}, g_1, c, \mu_0, \bar{p})$ with*

$$\begin{aligned}
&\mathbf{v} \in BC_w([0,\infty); L^2(\Omega)^d) \cap L^2(0,\infty; H^1_\nu(\Omega)),\\
&g_1 \in L^2(0,\infty; L^1_{(0)}(\Omega)), \qquad c \in BC_w([0,\infty); W^1_q(\Omega)),\\
&\mu_0 \in L^2(0,\infty; H^1(\Omega)), \qquad \bar{p} \in L^1_{loc}([0,\infty)),
\end{aligned}$$

where $H^1_\nu(\Omega) = \{\mathbf{v} \in H^1(\Omega)^d : \nu_{\partial\Omega}\cdot\mathbf{v}|_{\partial\Omega} = 0\}$, and such that $0 < \rho = \hat{\rho}(c) \in L^\infty(Q)$ is called a weak solution of (3.74)-(3.77), (3.69)-(3.71) if the following conditions are satisfied:

(i) For every $\boldsymbol{\varphi} \in C^\infty_0(0,\infty; H^1_\nu(\Omega)\cap L^\infty(\Omega)^d)$

$$\begin{aligned}
&-\int_0^\infty\int_\Omega P_\sigma\mathbf{v}\cdot\partial_t\boldsymbol{\varphi}\,dx\,dt + \int_0^\infty\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\varphi}\,dx\,dt\\
&\quad + \int_0^\infty\int_\Omega \rho^{-1}\mathbf{S}(c, D\mathbf{v}) : D\boldsymbol{\varphi}\,dx\,dt + \gamma\int_0^\infty\int_{\partial\Omega}\rho^{-1}\mathbf{v}_\tau\cdot\boldsymbol{\varphi}_\tau\,d\sigma\,dt\\
&\quad = \int_0^\infty\int_\Omega g_1\operatorname{div}\boldsymbol{\varphi}\,dx\,dt + \int_0^\infty\int_\Omega \big(\mu_0\nabla c + \nabla\rho^{-1}\cdot\mathbf{S}(c, D\mathbf{v})\big)\cdot\boldsymbol{\varphi}\,dx\,dt.
\end{aligned}$$

(ii) For every $\phi \in C^\infty_0(0,\infty; C^1(\overline{\Omega}))$

$$\begin{aligned}
\int_0^\infty\int_\Omega \rho\partial_t\phi\,dx\,dt + \int_0^\infty\int_\Omega \rho\mathbf{v}\cdot\nabla\phi\,dx\,dt &= 0,\\
\int_0^\infty\int_\Omega \rho c\partial_t\phi\,dx\,dt + \int_0^\infty\int_\Omega \rho c\mathbf{v}\cdot\nabla\phi\,dx\,dt &= \int_0^\infty\int_\Omega m(c)\nabla\mu\cdot\nabla\phi\,dx\,dt,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty\int_\Omega \rho^{-\frac{1}{q}}(\rho\mu_0 + \rho^2\bar{p} - \psi'(c))\phi\,dx\,dt = \beta\int_0^\infty\int_\Omega \rho^{-\frac{1}{q}}\rho^2 g_1\phi\,dx\,dt\\
&\quad - \beta\int_0^\infty\int_\Omega G(\mathbf{v})\partial_t\left(\rho(c)^{-\frac{1}{q}}\rho^2\phi\right)dx\,dt + \int_0^\infty\int_\Omega \rho^{1-\frac{1}{q}}|\nabla c|^{q-2}\nabla c\cdot\nabla\phi\,dx\,dt.
\end{aligned}$$

(iii) $(\mathbf{v}, c)|_{t=0} = (\mathbf{v}_0, c_0)$.

(iv) $(\mathbf{v}, c, \mu)$ *satisfy the energy inequality*

$$E(c(t), \mathbf{v}(t)) + \int_s^t \int_\Omega \left(\mathbf{S}(c, D\mathbf{v}) : D\mathbf{v} + m(c)|\nabla\mu|^2\right) dx\, d\tau$$
$$+ \gamma\|\mathbf{v}_\tau\|^2_{L^2(\partial\Omega\times(s,t))} \leq E(c(s), \mathbf{v}(s))$$

for all $t \in [s, \infty)$ *and almost all* $0 \leq s < \infty$ *including* $s = 0$.

Theorem 3.13. (Existence of Weak Solution, [4, Theorem 2.4]) *Let* $q > d$, $\delta, R > 0$. *Moreover, let* $\psi \in C^2(\mathbb{R})$, $\psi(c) \geq 0, \psi''(c) \geq -M$, *be given such that (3.73) holds. Then for every* $\mathbf{v}_0 \in L^2(\Omega)^d, c_0 \in W^1_q(\Omega)$ *with* $E(c_0, \mathbf{v}_0) \leq R$ *there exists a weak solution* $(\mathbf{v}, g_1, c, \mu_0, \bar{g})$ *of* (3.74)-(3.77),(3.69)-(3.71) *with the property that*

$$c(t, x) \in [-1-\delta, 1+\delta] \quad \text{for all } x \in \Omega, t \in (0, \infty),$$
$$g_1 \in L^2(0, \infty; L^p(\Omega)), \quad \bar{p} \in L^2_{loc}([0, \infty)).$$

The proof of Theorem 3.13 is based on a two-level approximation. First (3.74)-(3.77) is regularized by adding the terms $-\delta g_0 \frac{\mathbf{v}}{2}$ and δg_0 to the left-hand sides of (3.74) and (3.76), respectively. This gives an extra-term $-\delta \int_\Omega |g_0|^2\, dx$ on the right-hand side of (3.72). Existence of weak solutions for the regularized system is proved with the of a semi-implicit time discretization. Afterwards, one reformulates (3.74) as (3.83) together with the extra-term $-\delta g_0 \frac{\mathbf{v}}{2}$, derives suitable a priori estimate for g_0, g_1 and $\bar{g}$ and passes to the limit $\delta \to 0$.

Finally, let us comment on short time existence of strong solutions. In [7] the following result was shown:

Theorem 3.14 (Existence of Strong Solutions for Short Times, [7])**.** *Let* $\mathbf{v}_0 \in H^1_\nu(\Omega), c_0 \in H^2(\Omega)$ *with* $|c_0(x)| \leq 1$ *for all* $x \in \overline{\Omega}$ *and* $\nu_{\partial\Omega} \cdot \nabla c_0|_{\partial\Omega} = 0$, $d = 2, 3$, *and let the assumption throughout this subsection hold. Then there is some* $T > 0$ *such that there is a unique solution* $\mathbf{v} \in H^1(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^2(\Omega)^d)), c \in H^2(0, T; H^{-1}_{(0)}(\Omega)) \cap L^2(0, T; H^3(\Omega))$ *solving* (3.74)-(3.77),(3.69)-(3.71).

Here $H^{-1}_{(0)}(\Omega)$ is the dual of $H^1_{(0)}(\Omega) := H^1(\Omega) \cap L^2_{(0)}(\Omega)$.

The prove is based on a fixed-point argument and the unique solvability of the linearized system

$$\begin{aligned}
\partial_t \mathbf{v} - \operatorname{div} \widetilde{\mathbf{S}}(c_0, D\mathbf{v}) + \frac{\varepsilon}{\beta\alpha}\nabla \operatorname{div}(\rho^{-4}\nabla c') &= \mathbf{f}_1 && \text{in } \Omega \times (0, T), \quad (3.84)\\
\partial_t c' - \beta^{-1} \operatorname{div} \mathbf{v} &= f_2 && \text{in } \Omega \times (0, T),\\
(\nu_{\partial\Omega} \cdot \widetilde{\mathbf{S}}(c_0, D(P_\sigma \mathbf{v})))_\tau + \gamma(P_\sigma \mathbf{v})_\tau\Big|_{\partial\Omega} &= \mathbf{a} && \text{on } \partial\Omega \times (0, T),\\
\nu_{\partial\Omega} \cdot \mathbf{v}|_{\partial\Omega} = \nu_{\partial\Omega} \cdot \nabla c|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0, T),\\
(\mathbf{v}, c')|_{t=0} &= (\mathbf{v}_0, c'_0) && \text{in } \Omega,
\end{aligned}$$

where c' corresponds to ρc. To solve the latter system one uses the Helmholtz projection P_σ to decompose $\mathbf{v} = P_\sigma \mathbf{v} + \nabla G(\operatorname{div} \mathbf{v})$, where $(I - P_\sigma)\mathbf{v} = \nabla G(\operatorname{div} \mathbf{v})$. Moreover, P_σ and $I - P_\sigma$ are applied to (3.84). Throughout the analysis one has to solve a kind of damped plate equation of the form

$$\partial_t^2 c' - \Delta(a(c_0)\partial_t c') + \frac{\varepsilon}{\alpha\beta^2}\Delta \operatorname{div}(\rho_0^{-4}\nabla c') = f$$

up to lower order terms for some $a(c_0) > 0$. In order to solve this equation an abstract result by Chen and Triggiani [36] is applied. We note that the same kind of linearized system arises in the analysis of a Korteweg type model for compressible fluids with capillary stresses, cf. Kotschote [53]. Furthermore the linearized system differs very much from the linearized system of the model with same densities and the model with volume averaged densities.

4 Sharp Interface Limits

In this section it is shown in a formal way that the diffuse interface model of Abels, Garcke, Grün [12] (3.17)-(3.20) and the diffuse interface model (3.28)-(3.31) of Lowengrub and Truskinowsky both converge to the classical sharp interface model (2.1)-(2.5) if the parameter ε tends to zero. It was already noted in the introduction that the energy

$$\hat{\sigma} \int_\Omega \left(\frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi) \right) dx$$

converges to a multiple of the surface energy

$$\mathcal{H}^{d-1}(\Gamma)$$

where Γ denotes the sharp interface, see [60, 59]. One would hence expect that all terms involving $\hat{\sigma}$ will converge to terms involving interfacial energy and curvature, which is the first variation of interfacial energy. This will in fact be the case as one will see in the following analysis.

The method of formally matched asymptotic expansions, which is used in the following is based on the assumption that for small ε the domain Ω can at each time be separated into open subdomains $\Omega^\pm(t, \varepsilon)$ which are separated by a hypersurface $\Gamma(t, \varepsilon)$. In addition, it is assumed that the solutions have an asymptotic expansion in ε in the bulk regions away from $\Gamma(t, \varepsilon)$ and another suitable scaled expansion close to $\Gamma(t, \varepsilon)$. The scaling will be needed in the x–variable as the values of the phase field φ will change its value sharply but smoothly in a region of thickness ε. That leads to the formation of internal layers.

These expansions then have to be matched in a region where both expansions overlap. A detailed description of the method can be found in [44, 45, 9]. For some phase field models this approach can be justified rigorously, cf. [19, 38, 33, 10].

4.1 Models Based on a Volume Averaged Velocity

In this section the system

$$\partial_t(\rho(\varphi)\mathbf{v}) + \operatorname{div}(\mathbf{v} \otimes (\rho\mathbf{v} + \widetilde{\mathbf{J}})) - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \nabla p = \mu\nabla\varphi \,, \tag{4.1}$$

$$\operatorname{div}\mathbf{v} = 0 \,, \tag{4.2}$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \varepsilon m_0 \Delta\mu \,, \tag{4.3}$$

$$\frac{\hat{\sigma}}{\varepsilon}\psi'(\varphi) - \hat{\sigma}\varepsilon\Delta\varphi = \mu \,, \tag{4.4}$$

with

$$\widetilde{\mathbf{J}} = \frac{\tilde{\rho}_+ - \tilde{\rho}_-}{2}\mathbf{J}_\varphi = -\frac{\tilde{\rho}_+ - \tilde{\rho}_-}{2}\varepsilon m_0 \nabla\mu$$

and

$$\rho(\varphi) = \tilde{\rho}_+\frac{1+\varphi}{2} + \tilde{\rho}_-\frac{1-\varphi}{2}$$

is studied. This is basically the model (3.17)-(3.20) with the reformulation (3.23) and for simplicity $m = \varepsilon m_0$ is taken to be constant, see [9] for a more general case. We always denote the solution of (4.1)-(4.4) by $\mathbf{v}_\varepsilon, \widetilde{\mathbf{J}}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon$. In addition, we always assume that ψ is of double well form with two global minima at ± 1.

4.1.1 Outer Expansions

We assume that $\mathbf{v}_\varepsilon, \widetilde{\mathbf{J}}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon$ have in $\Omega(t,\varepsilon)$ an expansion of the form

$$u_\varepsilon(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \mathcal{O}(\varepsilon^2) \,.$$

Substituting these expansions into (4.1)-(4.4) leads to equations which have to be solved order by order.

The equation (4.4) gives to leading order ε^{-1}

$$\psi'(\varphi_0) = 0 \,.$$

The stable solutions of this equation are ± 1 and we denote $\Omega^\pm$ to be the sets where $\varphi_0 =$ and $\varphi_0 = -1$ respectively.

The expansions to order ε^0 of the fluid equations yield

$$\begin{aligned} \rho_\pm\partial_t\mathbf{v}_0 + \rho_\pm\mathbf{v}_0\cdot\nabla\mathbf{v}_0 - \eta_\pm\Delta\mathbf{v}_0 + \nabla p_0 &= 0 \,, \\ \operatorname{div}\mathbf{v}_0 &= 0 \end{aligned}$$

with the scaling chosen the equation (4.3) is fulfilled to leading order ε^0. However, we refer to [9] for the case when the mobility is scaled to be of order one.

4.1.2 Inner Expansions and Matching Conditions

It is now assumed that the zero level sets of $\varphi_\varepsilon(\cdot,t)$ converge for $\varepsilon \to 0$ to a smooth hypersurface $\Gamma(t)$ which moves with a normal velocity $\mathcal{V}$. As $\Gamma(t)$ is smooth one can define the signed distance function $d(x,t)$ of a point $x \in \Omega$ to $\Gamma(t)$ which is defined such that $d(x,t) > 0$ if $x \in \Omega_+(t)$ and negative if $x \in \Omega_-(t)$. Close to Γ the function d is smooth and we write each function $u(x,t)$ close to Γ in new coordinates $U(s,z,t)$ where s is a tangential spatial coordinate on Γ and $z(x,t) = d(x,t)/\varepsilon$. In the new coordinates the relevant differential operators transform as follows

$$\begin{aligned}
\partial_t u &= -\frac{1}{\varepsilon}\mathcal{V}\partial_z U + h.o.t. \\
\nabla_x u &= \frac{1}{\varepsilon}\partial_z U \boldsymbol{\nu} + \nabla_\Gamma U + h.o.t. \\
\Delta_x u &= \frac{1}{\varepsilon^2}\partial zz U - \frac{1}{\varepsilon} H \partial_z U - z|\mathbf{S}|^2 \partial_z U + \Delta_\Gamma U + h.o.t.\,,
\end{aligned}$$

where $\boldsymbol{\nu} = \nabla_x d$ is the unit normal pointing into $\Omega_+(t)$, ∇_Γ is the spatial surface gradient on Γ, $|\mathbf{S}|$ is the spectral norm of the Weingarten map $\mathbf{S}$, Δ_Γ is the Laplace–Beltrami operator on $\Gamma(t)$, and h.o.t. denotes terms of higher order in ε (see the Appendix of [9] for a proof).

Furthermore, it is assumed that the functions $\mathbf{v}_\varepsilon, p_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon$ as functions $(\mathbf{v}_\varepsilon, p_\varepsilon, \Phi_\varepsilon, M_\varepsilon)$ in the inner variables have an expansion of the form

$$u_\varepsilon(x,t) = U_\varepsilon(s,z,t) = U_0(t,s,z) + \varepsilon U_1(t,s,z) + \dots\,.$$

In an ε–dependent overlapping domain the outer and inner expansions have to coincide in a suitable sense when ε tends to zero. This leads to the following matching conditions which are derived in [44] and [45]. At a point $x \in \Gamma(t)$ with coordinate s it holds

$$\begin{aligned}
\lim_{z\to\pm\infty} U_0(s,z,t) &= u_0^\pm(x,t), \\
\lim_{z\to\pm\infty} \partial_z U_0(s,z,t) &= 0, \\
\lim_{z\to\pm\infty} \partial_z U_1(s,z,t) &= \nabla u_0^\pm(x,t)\cdot\boldsymbol{\nu},
\end{aligned}$$

where $u_0^\pm$ denotes the limit $\lim\limits_{\delta\to 0} u_0(x \pm \delta\boldsymbol{\nu})$ at a point $x \in \Gamma$.

4.1.3 Leading Order Equations

In the interfacial region the equation (4.4) gives to leading order $\frac{1}{\varepsilon}$:

$$\psi'(\Phi_0) - \partial_{zz}\Phi_0 = 0 \tag{4.5}$$

and matching with the outer solutions gives the following boundary condition at $\pm\infty$:

$$\lim_{z\to\pm\infty} \Phi_0(z) = \pm 1\,. \tag{4.6}$$

The problem (4.5), (4.6) has a unique solution with the property

$$\Phi_0(0) = 0\,,$$

see, e.g., [69], Section 2.6. This solution we choose in what follows. The equation $\operatorname{div}\mathbf{v} = 0$ gives to the order $\frac{1}{\varepsilon}$

$$\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu} = \partial_z(\mathbf{V}_0 \cdot \boldsymbol{\nu}) = 0$$

and together with the matching conditions we obtain that $\mathbf{V}_0 \cdot \boldsymbol{\nu}$ needs to be constant. We hence obtain

$$(\mathbf{v}_0^+ \cdot \boldsymbol{\nu})(x) = \lim_{z\to\infty}(\mathbf{V}_0 \cdot \boldsymbol{\nu})(z) = \lim_{z\to-\infty}(\mathbf{V}_0 \cdot \boldsymbol{\nu})(z) = (\mathbf{v}_0 \cdot \boldsymbol{\nu})(z)$$

and this gives

$$[\mathbf{v}_0 \cdot \boldsymbol{\nu}]_-^+ = 0\,.$$

At order $\frac{1}{\varepsilon}$ the diffusion type equation (4.3) leads to

$$-\mathcal{V}\partial_z\Phi_0 + (\mathbf{v}_0 \cdot \boldsymbol{\nu})\partial_z\Phi_0 = m_0\partial_{zz}M_0\,. \tag{4.7}$$

The matching conditions lead to $\partial_z M_0 \to 0$ and $\Phi_0(z) \to \pm 1$ for $z \to \pm\infty$. Hence (4.7) implies

$$\mathcal{V} = \mathbf{v}_0 \cdot \boldsymbol{\nu}$$

and

$$M_0 = M_0(s,t)\,.$$

In addition we obtain $[\mu]_-^+ = 0$ and hence $\mu^+ = \mu^- = M_0$. We now consider the momentum equation to leading order $\frac{1}{\varepsilon^2}$. Expressing $\nabla_x \mathbf{v}$ and $D_x\mathbf{v}$ in the new coordinates gives

$$\begin{aligned}
\nabla_x \mathbf{v} &= \frac{1}{\varepsilon}\partial_z \mathbf{v} \otimes \boldsymbol{\nu} + \nabla_\Gamma \mathbf{v} + h.o.t.\,,\\
D_x \mathbf{v} &= \frac{1}{2}\left[\frac{1}{\varepsilon}(\partial_z \mathbf{v} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \partial_z \mathbf{v}) + \frac{1}{2}(\nabla_\Gamma \mathbf{v} + (\nabla_\Gamma \mathbf{v})^T)\right] + h.o.t..
\end{aligned}$$

With the notation $\mathcal{E}(\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ we obtain

$$\begin{aligned}
\operatorname{div}_x \cdot (\eta(\varphi)D_x\mathbf{v}) &= \frac{1}{\varepsilon^2}\partial_z(\eta(\Phi)\mathcal{E}(\partial_z\mathbf{V} \otimes \boldsymbol{\nu}))\boldsymbol{\nu} + \frac{1}{\varepsilon}\partial_z(\eta(\Phi)\mathcal{E}(\nabla_\Gamma\mathbf{V}))\boldsymbol{\nu}\\
&\quad + \frac{1}{\varepsilon}\nabla_\Gamma \cdot (\eta(\Phi)\mathcal{E}(\partial_z\mathbf{V} \otimes \boldsymbol{\nu})) + \nabla_\Gamma \cdot (\eta(\Phi)\mathcal{E}(\nabla_\Gamma\mathbf{V})) + h.o.t..
\end{aligned}$$

Using $\partial_z\mathbf{V}_0 \cdot \boldsymbol{\nu} = 0$ we obtain

$$(\boldsymbol{\nu} \otimes \partial_z\mathbf{V}_0)\boldsymbol{\nu} = (\partial_z\mathbf{V}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu} = \mathbf{0}$$

and hence the momentum equation gives to leading order

$$\partial_z(\eta(\Phi_0)\partial_z\mathbf{V}_0) = 0\,.$$

The matching conditions imply that $\mathbf{V}_0$ is bounded and hence the above ODE only has constant solutions. The matching property $\lim_{z\to\pm\infty} \mathbf{v}_0(z) = \mathbf{v}^{\pm}(x)$ for $x \in \Gamma$ hence implies

$$[\mathbf{v}_0]_-^+ = 0\,.$$

4.1.4 Next Order Equations

The equation (4.4) which defines the chemical potential gives to the order ε^0

$$\hat{\sigma}\psi''(\Phi_0)\Phi_1 - \hat{\sigma}\partial_{zz}\Phi_1 = M_0 - \hat{\sigma}\partial_z\Phi_0 H\,. \tag{4.8}$$

As $\partial_z\Phi_0$ is in the kernel of the differential operator $u \mapsto \psi''(\Phi_0)u - \partial_{zz}u$ the right hand side of (4.8) needs to be L^2-orthogonal to $\partial_z\Phi_0$, see [9] for details on this Fredholm alternative type of argument. We hence obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \partial_z\Phi_0(M_0 - \hat{\sigma}\partial_z\Phi_0 H)dz \\ &= 2M_0 - \hat{\sigma}H\int_{-\infty}^{\infty} |\partial_z\Phi_0|^2 dz \\ &= 2\mu_0 - \sigma H \end{aligned}$$

where we set $\sigma = \hat{\sigma}c_0$ with

$$c_0 = \int_{-\infty}^{\infty} |\partial_z\Phi_0|^2\,.$$

It remains to derive the force balance (2.4) at the interface. We first observe that the term

$$\operatorname{div}(\mathbf{v}\otimes\widetilde{\mathbf{J}}) = -\left(\frac{\tilde{\rho}_+ - \tilde{\rho}_-}{2}\right)\varepsilon m_0 \operatorname{div}(\mathbf{v}\otimes\nabla\mu)$$

in the interfacial region gives no contribution to the order $\frac{1}{\varepsilon}$. Here one uses the facts that $\partial_z M_0 = 0$ and $\partial_z\mathbf{v}_0 = 0$. One hence obtains that (4.1) to order $\frac{1}{\varepsilon}$ gives the identity

$$\begin{aligned} -\partial_z(\rho(\Phi_0)\mathbf{V}_0)\mathcal{V} + \partial_z(\rho(\Phi_0)(\mathbf{V}_0\otimes\mathbf{V}_0))\boldsymbol{\nu} - 2\partial_z(\eta(\Phi_0)\mathcal{E}(\partial_z\mathbf{V}_1\otimes\boldsymbol{\nu})\boldsymbol{\nu}) \\ -2\partial_z(\eta(\Phi_0)\mathcal{E}(\nabla_\Gamma\mathbf{V}_0)\boldsymbol{\nu}) + \partial_z P_0\boldsymbol{\nu} = \mu\partial_z\Phi_0\boldsymbol{\nu}\,. \end{aligned} \tag{4.9}$$

The matching conditions require $\lim_{z\to\pm\infty}\partial_z\mathbf{V}_1(z) = \nabla\mathbf{v}_0^{\pm}(x)\nu$ and hence

$$\partial_z\mathbf{V}_1\otimes\nu + \nabla_\Gamma\mathbf{V}_0 \to \nabla_x\mathbf{v}_0 \text{ for } z\to\pm\infty\,.$$

Integrating (4.9) with respect to z now gives

$$-[\rho_0\mathbf{v}_0]_-^+\mathcal{V} + [\rho_0\mathbf{v}_0]_-^+\mathbf{v}_0\cdot\boldsymbol{\nu} - 2[\eta\varepsilon(\nabla_x\mathbf{v}_0)]_-^+\boldsymbol{\nu} = \hat{\sigma}\left(\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2 dz\right)H\boldsymbol{\nu} + [p_0]_-^+\boldsymbol{\nu}\,.$$

The identity $\mathcal{V} = \mathbf{v}_0 \boldsymbol{\nu}$ hence gives

$$-2[\eta D\mathbf{v}_0]_-^+ \boldsymbol{\nu} + [p_0]_-^+ \boldsymbol{\nu} = \sigma H \boldsymbol{\nu}\,.$$

We hence obtained all equations which appeared in the sharp interface problem (2.1)-(2.5).

4.1.5 The Navier–Stokes/Mullins–Sekerka System as Sharp Interface Limit

It is also possible to obtain the Navier–Stokes/Mullins–Sekerka system (2.23)-(2.29) as sharp interface limit. To achieve this, one has to use a different scaling in (4.3). In fact (4.3) is replaced by

$$\partial_t \varphi + \mathbf{v} \cdot \nabla \varphi = 2m\Delta\mu\,. \tag{4.10}$$

Expansions in $\Omega_\pm$ immediately give

$$\Delta\mu_0 = 0\,.$$

At order $\frac{1}{\varepsilon}$ one obtains from (4.10) that

$$(-\mathcal{V} + \mathbf{v}_0 \cdot \boldsymbol{\nu})\partial_z \Phi_0 = 2m\partial_{zz} M_1\,.$$

Matching requires $\partial_z M_1 \to \nabla\mu_0 \cdot \boldsymbol{\nu}$ and integration of the above equation gives

$$(-\mathcal{V} + \mathbf{v}_0 \cdot \boldsymbol{\nu}) = m[\nabla\mu_0 \cdot \boldsymbol{\nu}]_-^+\,,$$

which is precisely equation (2.29).

4.2 Sharp Interface Expansions for the Lowengrub–Truskinovsky Model

We now consider the sharp interface limit of the Lowengrub–Truskinovsky model

$$\partial_t \rho(\mathbf{v}) + \operatorname{div}(\rho\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S}(c, D\mathbf{v}) + \nabla p = \hat{\sigma}\varepsilon \operatorname{div}(\nabla c \otimes \nabla c)\,, \tag{4.11}$$

$$\rho(c)(\partial_t c + \nabla c \cdot \mathbf{v}) = \hat{m}\varepsilon^2 \Delta\mu\,, \tag{4.12}$$

$$\operatorname{div} \mathbf{v} = \beta\hat{m}\varepsilon^2 \Delta\mu\,, \tag{4.13}$$

$$\frac{\hat{\sigma}}{\varepsilon}\psi'(c) - \hat{\sigma}\varepsilon\Delta c - \beta\rho(c)p = \rho(c)\mu\,, \tag{4.14}$$

where $\mathbf{S}(c, D\mathbf{v}) = 2\eta(c)D(\mathbf{v}) + \lambda(c) \operatorname{div} \mathbf{v}\, \mathrm{Id}$ and we have set $m = \varepsilon^2 \hat{m}$ and assume that the functional relation between ρ and c is of simple mixture type, see (3.26).

4.2.1 Outer Expansions

In the phases $\Omega^{\pm}$ we obtain as in the preceding section

$$c_0 = \pm 1 \ , \ \ \rho_0 = \rho_{\pm}$$

and hence

$$\operatorname{div} \mathbf{v}_0 = 0 \, .$$

This then implies

$$\rho_{\pm}\partial_t \mathbf{v}_0 + \rho_{\pm}\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 - \eta_{\pm}\Delta \mathbf{v}_0 + \nabla p_0 = 0 \, .$$

4.2.2 Inner Expansion to Leading Order

The expansions in the interface are as in the case of the volume averaged velocity with the two exceptions

$$\begin{aligned} c_\varepsilon &= C_\varepsilon(s,z,t) = C_0(t,s,z) + \varepsilon C_1(t,s,z) + \dots \, , \\ p_\varepsilon &= P_\varepsilon(s,z,t) = \varepsilon^{-1} P_{-1}(t,s,z) + P_0(t,s,z) + \dots \, . \end{aligned}$$

In the interface the term $\varepsilon \nabla c \otimes \nabla c$ will give a contribution to the order ε^{-1} which has to be balanced by the pressure. This is due to the fact that in contrast to the volume averaged case we do not work with $\mu \nabla c$ as a capillarity term. Therefore, the inner expansion of the pressure has P_{-1} as the leading order term.

For the capillarity type term $\varepsilon \operatorname{div}(\nabla c \otimes \nabla c)$ we obtain

$$\begin{aligned} \varepsilon \operatorname{div}(\nabla c \otimes \nabla c) &= \frac{1}{\varepsilon^2}\partial_z(|\partial_z C|^2 \boldsymbol{\nu}) + \frac{1}{\varepsilon}(\partial_z C \nabla_\Gamma C) \\ &\quad + \frac{1}{\varepsilon}\nabla_\Gamma \cdot (|\partial_z C|^2 \boldsymbol{\nu} \otimes \boldsymbol{\nu}) + \nabla_\Gamma \cdot (\partial_z C(\boldsymbol{\nu} \otimes \nabla_\Gamma c + \nabla_\Gamma c \otimes \boldsymbol{\nu})), \\ &\quad + h.o.t. \end{aligned} \tag{4.15}$$

where we already notice the $\frac{1}{\varepsilon^2}$ contribution of this term in the momentum balance. Similar as in the previous section we obtain from (4.13) that $\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu} = 0$ which leads to

$$[\mathbf{v}_0] \cdot \boldsymbol{\nu} = 0 \, .$$

The equation (4.12) gives to leading order

$$\rho(C_0)\partial_z C_0(-\mathcal{V} + \mathbf{V}_0 \cdot \boldsymbol{\nu}) = 0 \, .$$

Matching implies

$$\lim_{z \to \pm\infty} C_0(z) = \pm 1$$

which implies $\partial_z C_0 \not\equiv 0$ and hence

$$\mathcal{V} = \mathbf{v}_0 \cdot \boldsymbol{\nu} \, .$$

The momentum balance (4.11) gives to leading order ε^{-2}

$$-\partial_z(\eta(C_0)\partial_z \mathbf{V}_0) + \partial_z P_{-1}\boldsymbol{\nu} = -\hat{\sigma}\partial_z|\partial_z C_0|^2\boldsymbol{\nu}\,. \tag{4.16}$$

Since $\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu} = 0$ we obtain from the normal part of the above equation

$$P_{-1}(t,s,z) = \hat{P}(t,s) - \hat{\sigma}|\partial_z C_0|^2(t,s,z)\,.$$

Matching requires $P_{-1} \to 0$ and $\partial_z C_0 \to 0$ for $z \to \pm\infty$ and hence $\hat{P} \equiv 0$ which gives

$$P_{-1} = -\hat{\sigma}|\partial_z C_0|^2\,.$$

Hence (4.16) boils down to

$$\partial_z(\eta(C_0)\partial_z \mathbf{V}_0) = 0$$

which implies after matching

$$[\mathbf{v}_0] = 0\,.$$

The equation (4.14) gives to leading order ε^{-1}

$$\hat{\sigma}\psi'(C_0) - \hat{\sigma}\partial_{zz}C_0 - \beta\rho(C_0)P_{-1} = 0\,.$$

Using $\beta = -\rho'/\rho^2$ and $P_{-1} = -\hat{\sigma}|\partial_z C_0|^2$ gives

$$\hat{\sigma}\psi'(C_0) - \frac{\hat{\sigma}}{\rho}(\partial_z(\rho\partial_z C_0)) = 0\,.$$

This ODE has a unique solution fulfilling $C_0(\pm\infty) = \pm 1$ and $C_0(0) = 0$. In particular C_0 is independent of s and t.

4.2.3 Inner Expansions to Next Leading Order

Using (4.15) and $\nabla_\Gamma C_0 \equiv 0$ we obtain from the momentum balance (4.11) to order ε^{-1}

$$\begin{aligned}&-2\partial_z(\eta(C_0)\mathcal{E}(\partial_z \mathbf{V}_1 \otimes \boldsymbol{\nu})\boldsymbol{\nu}) - 2\partial_z(\eta(C_0)\mathcal{E}(\nabla_\Gamma \mathbf{V}_0)\boldsymbol{\nu})\\ &\quad + \partial_z P_0\hat{\sigma}\boldsymbol{\nu} + \nabla_\Gamma P_{-1} + \hat{\sigma}\partial_z(2\partial_z C_0\partial_z C_1\boldsymbol{\nu}) + \hat{\sigma}\nabla_\Gamma\cdot(|\partial_z C_0|^2\boldsymbol{\nu}\otimes\boldsymbol{\nu}) = 0\end{aligned}$$

where as above we used $\mathcal{V} = \mathbf{v}_0\cdot\boldsymbol{\nu}$ which yields that the kinetic term gives no contribution. Since C_0 is independent of s and t, we obtain that $\nabla_\Gamma P_{-1} \equiv 0$. We also compute

$$\nabla_\Gamma\cdot(|\partial_z C_0|^2\boldsymbol{\nu}\otimes\boldsymbol{\nu}) \;=\; |\partial_z C_0|^2\nabla_\Gamma\cdot(\boldsymbol{\nu}\otimes\boldsymbol{\nu}) = -H|\partial_z C_0|^2\boldsymbol{\nu}\,.$$

Hence we obtain

$$\begin{aligned}&-2\partial_z(\eta(C_0)\mathcal{E}(\partial_z \mathbf{V}_1 \otimes \boldsymbol{\nu})\boldsymbol{\nu}) - 2\partial_z(\eta(C_0)\mathcal{E}(\nabla_\Gamma \mathbf{V}_0)\boldsymbol{\nu})\\ &\qquad + \partial_z P_0\boldsymbol{\nu} + \hat{\sigma}\partial_z(2\partial_z C_0\partial_z C_1\boldsymbol{\nu}) - \hat{\sigma}H|\partial_z C_0|^2\boldsymbol{\nu} = 0\,.\end{aligned}$$

Integrating and using the matching conditions gives similar as in Section 4.1.4

$$-2[\eta D\mathbf{v}_0]_-^+\boldsymbol{\nu} + [P_0]_-^+\boldsymbol{\nu} = \sigma H\boldsymbol{\nu}\,,$$

where we used that $\int_{-\infty}^{\infty} \partial_z(\partial_z C_0 \partial_z C_1)dz = 0$ which follows from matching. We hence obtain that also the Lowengrub–Truskinovsky model yields the sharp interface model (2.1)-(2.5) in the asymptotic limit $\varepsilon \to 0$.

4.3 Known Results on Sharp Interface Limits

First results on the sharp interface limits of Cahn–Hilliard/Navier–Stokes systems are for a simplified situation due to Lowengrub and Truskinovsky [58]. They used the method of formally matched asymptotic expansions. In the general case the sharp interface limit has been analyzed with formally matched asymptotic expansions by Abels, Garcke and Grün [12] where also different scalings have been analyzed which lead to quite different asymptotic limits.

So far only very few rigorous results for the sharp interface limit exist. Abels and Röger [16] and Abels and Lengeler [14] showed convergence in the sense of varifold solutions, cf. Chen [35], for the case in which m is constant. Abels and Röger [16] studied the case of matched densities and m independent of ε. Abels and Lengeler [14] considered the case of a volume averaged velocity and m independent of ε as well as $m = m(\varepsilon) \to_{\varepsilon\to 0} 0$ sublinearly, i.e., $\frac{\varepsilon}{m(\varepsilon)} \to_{\varepsilon\to 0} 0$. Moreover, it is shown that certain radially symmetric solutions of (3.55)-(3.58) tend to functions which will not satisfy the Young–Laplace law (2.5) in the limit $\varepsilon \to 0$ if the mobility tends to zero faster than ε^3. A result on a sharp interface limit to solutions which fulfill the limit equations in a stronger sense is still open.

Finally, let us note that Abels and Schaubeck [17] showed that for mobilities m tending to zero faster than ε^3 in the convective Cahn–Hilliard equation, i.e., (3.58), (3.59) with a given velocity smooth and solenoidal field $\mathbf{v}$, the surface tension term $-\varepsilon\,\mathrm{div}(\nabla\varphi_\varepsilon \otimes \nabla\varphi_\varepsilon)$ in general does not converge to a multiple of the mean curvature vector as ε tends to zero. For a related Allen–Cahn/Stokes system Abels and Liu [15] are able to show converge to solutions which fulfill the sharp interface problem in a strong sense for small times.

5 Conclusions

Because of possible singularities in the interface, the mathematical description of a two-phase of macroscopically immiscible fluids remains a mathematical challenge with many open problems and questions. We have discussed weak formulations of the classical sharp interface model for two viscous, incompressible, immiscible Newtonian fluids. In the absense of surface tension, existence of weak solutions is known, but their is little control of the regularity of the interface known. In particular, it cannot be excluded that it is dense in the domain in general. In the case with surface tension, the energy estimates provides a control of the total surface measure of the interface. But existence of weak solutions is unknown since possible oscillation and concentration effects of the interface prevent from passing to the limit in the weak formulation of the mean curvature vector, which arises due to the Young-Laplace law. Moreover, we discussed a non-classical sharp interface model, where the classical kinematic condition that the interface is transported by the fluid velocity is replaced by a convective Mullins-Sekerka equation. This model arises as the sharp interface limit of a diffuse interface model if the mobility coefficient in the diffuse interface model does not tend to zero. For this model existence of weak solutions can be shown with similar techniques as for the Mullins-Sekerka system since an additional term in the energy inequality gives rise to a suitable a priori bound of the mean curvature of the interface.

In order to describe two-phase flows beyond the occurance of topological singularities, diffuse interface models, where the macroscopically immiscible fluids are considered as partly miscible, are an important alternative. In these models the sharp interface and the charakteristic function of one phase is replaced by an order parameter, which varies smoothly, but with a steep gradient in a thin interfacial region. In the case of different densities, there are different models in dependence of choice of the mean velocity for the fluid mixture. The choice of a volume averaged velocity leads to a divergence free velocity field and a system, which is very similar to the case of same densities. We discussed results on existence of weak solutions for different choices of the free energy and mobility. For a barycentric/mass averaged velocity the velocity field is no longer divergence free and the pressue enters the equation for the chemical potential. This leads to significant new difficulties in the mathematical analysis of this model. Moreover, the linearized system is rather different from the case of same densities. In this case existence of weak solutions is only known in the case of a free energy, which is non-quadratic in the gradient of the concentration.

Finally, we discussed the sharp interface limit of the diffuse interface models to the classical sharp interface model. This convergence can be discussed using the method of formally matched asymptotic expansions. But there are only few mathematical rigorous convergence results. In particular a proof of convergence to strong solutions of the limit equations remains an open problem, even for small times.

6 Cross-References

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