Lecture 4

Type II Singularities

We assume now that we are in the type II singularity case, that is,

$$\limsup_{t\to T} \max_{p\in M} |\mathrm{A}(p,t)|\sqrt{T-t} = +\infty$$

for the mean curvature flow of a compact hypersurface $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ in its maximal interval of existence.

A good question is actually whether type II singularities there exist.

An example is given by a closed, symmetric, self–intersecting curve with the shape of a symmetric "eight" figure in the plane, which has zero *rotation number*. Pushing a little the analysis of the previous lecture and keeping into account the symmetries of the curve, if the curve develops a type I singularity, we can produce a nonflat blow up limit which is homothetic and nonflat. Then such a limit must be a circle or one of Abresch–Langer curves. In both cases, the limit would be a compact closed curve and by the smooth convergence, the rotation number would still be zero. Hence, the circle has to be excluded and the contradiction is given by the fact that there are no Abresch–Langer curves with zero rotation number. Hence, type I singularities do not describe all the possible ones.

Another example is given by a cardioid–like curve in the plane with a very small loop, hence high curvature: one can right guess that at some time the loop shrinks while the rest of the curve remains smooth and a cusp develops. Such a singularity is of type II, since if we have a type I singularity we would get an Abresch–Langer curve as a blow up limit and this implies, as these latter are compact, that the entire curve has vanished in a single point (see the analysis in [15] and also [14, 16]).

As we will see in Theorem 4.5.5 that embedded curves do not develop type II singularities, one could reasonably conjecture that also for embedded hypersurfaces (at least in low dimension) all the singularities are of type I. Unfortunately, this is not true even if the dimension is only two, indeed, the following example excludes such a good behavior.

Example (The Degenerate Neckpinch). For a given $\lambda > 0$, let us set

$$\phi_\lambda(x) = \sqrt{(1-x^2)(x^2+\lambda)}, \qquad -1 \leq x \leq 1.$$

For any $n \geq 2$, let M^λ be the n–dimensional hypersurface in $\mathbb{R}^{n+1}$ obtained by rotation of the graph of ϕ_λ in $\mathbb{R}^2$. The hypersurface M^λ looks like a dumbbell, where the parameter λ measures the width of the central part. Then, it is possible to prove the following properties (see [4]):

1. if λ is large enough, the hypersurface M^λ_t eventually becomes convex and shrinks to a point in finite time;

2. if λ is small enough, M^λ_t exhibits a neckpinch singularity as in the case of the *standard neckpinch* (see Section 1.4);

3. there exists at least one intermediate value of $\lambda > 0$ such that M_t^λ shrinks to a point in finite time, has positive mean curvature up to the singular time, but never becomes convex. The maximum of the curvature is attained at the two points where the surface meets the axis of rotation;

4. in this latter case the singularity is of type II, otherwise the blow up at the singular time would give a sphere (for all $p \in M$ we would have $\widehat{p} = O \in \mathbb{R}^{n+1}$ hence, by estimate (3.2.2), any limit hypersurface is bounded). This is impossible as it would imply that the surface would have been convex at some time.

The flowing hypersurface at point (3) is called the *degenerate neckpinch* and was first conjectured by Hamilton for the Ricci flow [61, Section 3]. Intuitively speaking, it is a limiting case of the neckpinch where the cylinder in the middle and the two spheres on the sides shrink at the same time. One can also build the example in an asymmetric way, with only one of the two spheres shrinking simultaneously with the neck, while the other one remains nonsingular.
A sharp analysis of the singular behavior for a class of rotationally symmetric surfaces exhibiting a degenerate neckpinch has been done by Angenent and Velázquez in [19].
Another interesting example of singularity formation (a family of evolving tori, proposed by De Giorgi) was carefully studied by Soner and Souganidis in [110, Proposition 3] (see also the numerical analysis performed by Paolini and Verdi in [101, Section 7.5]).

4.1 Hamilton's Blow Up

In order to deal with the blow up around type II singularities we need a new set of estimates which are actually independent of the type II hypothesis and scaling invariant (see [3] and [104]).

Proposition 4.1.1. *Let $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact hypersurface such that $\sup_{p\in M} |\mathrm{A}(p, 0)| \leq \Lambda < +\infty$. Then, there exists a time $\tau = \tau(\Lambda) > 0$ and constants $C_m = C_m(\Lambda)$, for every $m \in \mathbb{N}$ such that $|\nabla^m \mathrm{A}(p, t)|^2 \leq C_m/t^m$ for every $p \in M$ and $t \in (0, \tau)$.*

Proof. We prove the claim by induction. By the evolution equation for $|\mathrm{A}|^2$,

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4 \leq \Delta|\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

we get

$$\frac{\partial}{\partial t}|\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,,$$

hence, there exists a time $\tau = \tau(\Lambda) > 0$ and a constant $C_0 = C_0(\Lambda)$ such that $|\mathrm{A}(p, t)|^2 \leq C_0$ for every $p \in M$ and $t \in [0, \tau)$. This is the case $m = 0$.
Recalling equation (2.3.5), setting $f = \sum_{k=0}^{m} |\nabla^k \mathrm{A}|^2 \lambda_k t^k$ for some positive constants $\lambda_0, \dots, \lambda_m$ and assuming the inductive hypothesis $|\nabla^k \mathrm{A}(p, t)|^2 \leq C_k(\Lambda)/t^k$ for any $k \in \{0, \dots, m-1\}$, $p \in M$

and $t \in (0, \tau)$, we compute

$$\begin{aligned}
\frac{\partial}{\partial t} f &= \frac{\partial}{\partial t} \sum_{k=0}^{m} |\nabla^k \mathrm{A}|^2 \lambda_k t^k \\
&= \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 k \lambda_k t^{k-1} \\
&\quad + \sum_{k=0}^{m} \lambda_k t^k \Big(\Delta |\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r \in \mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A} \Big) \\
&\leq \Delta f + \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} - 2|\nabla^{m+1}\mathrm{A}|^2 \lambda_m t^m \\
&\quad + \sum_{k=0}^{m} \lambda_k t^k C(k) \sum_{p+q+r=k \,|\, p,q,r \in \mathbb{N}} |\nabla^p \mathrm{A}||\nabla^q \mathrm{A}||\nabla^r \mathrm{A}||\nabla^k \mathrm{A}| \\
&\leq \Delta f + \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} + \sum_{k=0}^{m-1} \lambda_k C(k) \sum_{p+q+r=k \,|\, p,q,r \in \mathbb{N}} C_p C_q C_r C_k \\
&\quad + \lambda_m t^{m/2} C(m) \Big(\sum_{p+q+r=m \,|\, p,q,r<m} C_p C_q C_r \Big) |\nabla^m \mathrm{A}| + \lambda_m t^m C(m) |\mathrm{A}|^2 |\nabla^m \mathrm{A}|^2 \\
&\leq \Delta f + \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} + C\lambda_m t^m |\nabla^m \mathrm{A}|^2 + D
\end{aligned}$$

where in the last passage we applied Peter–Paul inequality. If we choose now inductively positive constants $\lambda_1, \dots, \lambda_m$ such that $\lambda_k = 2\lambda_{k-1}/k$ starting with $\lambda_0 = 1$ (easily $\lambda_k = 2^k/k!$), we have

$$\frac{\partial}{\partial t} f \leq \Delta f + C\lambda_m t^m |\nabla^m \mathrm{A}|^2 + D \leq \Delta f + Cf + D\,,$$

for every $p \in M$ and $t \in (0, \tau)$, where the constants C and D depend only on m and Λ, by the inductive hypothesis. Notice that the inequality holds also at $t = 0$ as the function f is smooth on $M \times [0, \tau)$.
This differential inequality, by the maximum principle, then implies that $f_{\max}(t)$ is bounded in the interval $[0, \tau)$ by some constant C depending only on m, Λ and $f_{\max}(0) = |\mathrm{A}|^2_{\max}(0) \leq \Lambda^2$, hence

$$t^m |\nabla^m \mathrm{A}(p,t)|^2 \leq f(t)/\lambda_m \leq C/\lambda_m = C_m$$

in the interval $t \in [0, \tau)$, then we are done as $C_m = C_m(\Lambda)$. □

The following corollary is an easy consequence.

Corollary 4.1.2. *Let $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact hypersurface such that $\sup_{p \in M} |\mathrm{A}(p, 0)| \leq \Lambda < +\infty$. Then, there exists a value $\tau = \tau(\Lambda) > 0$ and constants C_m for every $m \in \mathbb{N}$, depending only on Λ such that $|\nabla^m \mathrm{A}(p,t)|^2 \leq C_m$ for every $p \in M$ and $t \in (\tau/2, \tau)$.*
For instance, one can choose $\tau = 1/(4\Lambda^2)$.

Proof. Only the last claim needs an explanation, it follows by integrating the differential inequality

$$\frac{\partial}{\partial t} |\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,.$$

□

Remark 4.1.3. These estimates provide another proof of Proposition 2.4.8, moreover they can replace the estimates of Proposition 3.2.9 in the proof of Proposition 3.2.10.

We describe now Hamilton's procedure to get a blow up flow at a type II singularity of the mean curvature flow of a compact hypersurface at time $T > 0$.

Let us choose a sequence of times $t_k \in [0, T - 1/k]$ and points $p_k \in M$ such that

$$|\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k) = \max_{\substack{t\in[0,T-1/k]\\ p\in M}} |\mathrm{A}(p, t)|^2(T - 1/k - t)\,. \tag{4.1.1}$$

This maximum goes to $+\infty$ as $k \to \infty$, indeed, if it is bounded by some constant C on a subsequence $k_i \to \infty$, for every $t \in [0, T)$ definitely we have $t \in [0, T - 1/k_i]$ and

$$|\mathrm{A}(p, t)|^2(T - t) = \lim_{i\to\infty} |\mathrm{A}(p, t)|^2(T - 1/k_i - t) \leq C$$

for every $p \in M$. This is in contradiction with the type II condition

$$\limsup_{t\to T} \max_{p\in M} |\mathrm{A}(p, t)|\sqrt{T - t} = +\infty\,.$$

This fact also forces the sequence t_k to converge to T as $k \to \infty$. If t_{k_i} is a subsequence not converging to T, we would have that the sequence $|\mathrm{A}(p_{k_i}, t_{k_i})|^2$ is bounded, hence also

$$\max_{\substack{t\in[0,T-1/k_i]\\ p\in M}} |\mathrm{A}(p, t)|^2(T - 1/k_i - t)$$

would be bounded.

Thus, we can choose an increasing (not relabeled) subsequence t_k converging to T, such that $|\mathrm{A}(p_k, t_k)|$ goes monotonically to $+\infty$ and

$$|\mathrm{A}(p_k, t_k)|^2 t_k \to +\infty\,, \qquad |\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k) \to +\infty\,,$$

Moreover, we can also assume that $p_k \to p$ for some $p \in M$.

We rescale now *the flow* as follows: let $\varphi_k : M \times I_k \to \mathbb{R}^{n+1}$, where

$$I_k = [-|\mathrm{A}(p_k, t_k)|^2 t_k, |\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k)]\,,$$

be the evolution given by

$$\varphi_k(p, s) = |\mathrm{A}(p_k, t_k)|[\varphi(p, s/|\mathrm{A}(p_k, t_k)|^2 + t_k) - \varphi(p_k, t_k)]$$

and we set $M^k_s = \varphi_k(M, s)$ and A_k the second fundamental form of the flowing hypersurfaces φ_k.

It is easy to check that this is a parabolic rescaling hence, every φ_k is still a mean curvature flow, moreover the following properties hold

- $\varphi_k(p_k, 0) = 0 \in \mathbb{R}^{n+1}$ and $|\mathrm{A}_k(p_k, 0)| = 1$,
- for every $\varepsilon > 0$ and $\omega > 0$ there exists $\overline{k} \in \mathbb{N}$ such that

$$\max_{p\in M} |\mathrm{A}_k(p, s)| \leq 1 + \varepsilon \tag{4.1.2}$$

  for every $k \geq \overline{k}$ and $s \in [-|\mathrm{A}(p_k, t_k)|^2 t_k, \omega]$.

Indeed, (the first point is immediate), by the choice of the minimizing pairs (p_k, t_k) we get

$$\begin{aligned}
|\mathrm{A}_k(p, s)|^2 &= |\mathrm{A}(p_k, t_k)|^{-2}|\mathrm{A}(p, s/|\mathrm{A}(p_k, t_k)|^2 + t_k)|^2\\
&\leq |\mathrm{A}(p_k, t_k)|^{-2}|\mathrm{A}(p_k, t_k)|^2 \frac{T - 1/k - t_k}{T - 1/k - t_k - s/|\mathrm{A}(p_k, t_k)|^2}\\
&= \frac{|\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k)}{|\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k) - s}\,,
\end{aligned}$$

if $s/|\mathrm{A}(p_k, t_t)|^2 + t_k \in [0, T - 1/k]$, that is, if $s \in I_k$. Then, assuming $s \leq \omega$ and k large enough, the claim follows as we know that $|\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k) \to +\infty$.

This discussion implies that if we are able to take a (subsequential) limit of these *flows*, locally smoothly converging in every compact time interval, we would get a mean curvature flow such that the norm of the second fundamental form is uniformly bounded by one and the time interval of existence is the whole $\mathbb{R}$ as $\lim_{k\to\infty} I_k = (-\infty, +\infty)$.
This is ensured by the next proposition.

Proposition 4.1.4. *The family of flows φ_k converges (up to a subsequence) in the C^∞_{loc} topology to a nonempty, smooth evolution by mean curvature of complete hypersurfaces M^∞_s in the time interval $(-\infty, +\infty)$. Such a flow is called* eternal, *as a consequence it cannot contain compact hypersurfaces.*
Moreover, the second fundamental form and all its covariant derivatives are uniformly bounded and $|\mathrm{A}_\infty|$ takes its absolute maximum, which is 1, at time $s = 0$ at the origin of $\mathbb{R}^{n+1}$, hence the limit flow is nonflat. Finally, if the original initial hypersurface was embedded this limit flow consists of embedded hypersurfaces.

Proof. By the previous discussion, in every bounded interval of time $[s_1, s_2]$ the evolutions φ_k have definitely uniformly bounded curvature, precisely $|\mathrm{A}_k| \leq (1 + \varepsilon)$, then for $\varepsilon << 1$ by Corollary 4.1.2 in every interval $[s_1 + 1/16, s_1 + 1/8]$ we have uniform estimates $|\nabla^m \mathrm{A}_k| \leq C_m$ with C_m independent of s_1, for every $m \in \mathbb{N}$.
By means of the monotonicity formula we can have a uniform estimate on $\widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R)$ as follows (we recall that $\widetilde{\mathcal{H}}^n$ is the n–dimensional Hausdorff measure counting multiplicities): we set μ^k_s to be the measure associated to the hypersurface φ_k at time s and μ_0 the measure associated to the initial hypersurface φ_0, then

$$\begin{aligned}
\widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R) &= \int_M \chi_{B_R}(y)\, d\mu^k_s(y) \\
&\leq \int_M \chi_{B_R}(y) e^{\frac{R^2 - |y|^2}{4}}\, d\mu^k_s(y) \\
&\leq e^{R^2/4} \int_M e^{-\frac{|y|^2}{4}}\, d\mu^k_s(y) \\
&= (4\pi)^{n/2} e^{R^2/4} \int_M \frac{e^{-\frac{|y|^2}{4(s+1-s)}}}{[4\pi(s + 1 - s)]^{n/2}}\, d\mu^k_s(y) \\
&\leq C(R) \int_M \frac{e^{-\frac{|y|^2}{4(s+1+|\mathrm{A}(p_k,t_k)|^2 t_k)}}}{[4\pi(s + 1 + |\mathrm{A}(p_k, t_k)|^2 t_k)]^{n/2}}\, d\mu^k_{-|\mathrm{A}(p_k,t_k)|^2 t_k}(y) \\
&= C(R) \int_M \frac{|\mathrm{A}(p_k, t_k)|^n e^{-\frac{|x-\varphi(p_k,t_k)|^2 |\mathrm{A}(p_k,t_k)|^2}{4(s+1+|\mathrm{A}(p_k,t_k)|^2 t_k)}}}{[4\pi(s + 1 + |\mathrm{A}(p_k, t_k)|^2 t_k)]^{n/2}}\, d\mu_0(x) \\
&\leq C(R) \int_M \frac{|\mathrm{A}(p_k, t_k)|^n}{[4\pi(s + 1 + |\mathrm{A}(p_k, t_k)|^2 t_k)]^{n/2}}\, d\mu_0(x) \\
&\leq C(R) \mathrm{Area}(\varphi_0) \frac{|\mathrm{A}(p_k, t_k)|^n}{[4\pi(s + 1 + |\mathrm{A}(p_k, t_k)|^2 t_k)]^{n/2}}\,,
\end{aligned}$$

hence, if s stays in a bounded interval $[s_1, s_2] \subset \mathbb{R}$, we have

$$\limsup_{k\to\infty} \widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R) \leq C(R) \frac{\mathrm{Area}(\varphi_0)}{[4\pi T]^{n/2}} = C(R, \varphi_0)\,.$$

This implies that

$$\widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R) \leq C(R, \varphi_0, s_1, s_2)$$

uniformly in $s \in [s_1, s_2]$ and where the constant C is independent of $k \in \mathbb{N}$.
Then we use the same argument of Proposition 3.2.10, but applied to *flows*, that is, we consider

the time–tracks of the flows φ_k as hypersurfaces $\widetilde{\varphi}_k : M \times I_k \to \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$ defined by $\widetilde{\varphi}_k(p,s) = (\varphi_k(p,s), s)$ and we reparametrize them locally as graphs of smooth functions.
Reasoning like in the proof of Proposition 2.4.9, the estimates on the space covariant derivatives of A_k imply uniform locally estimates on space and also time derivatives (using the evolution equation) of the representing functions, so up to a subsequence we can get locally a limit smooth mean curvature flow. By a diagonal argument, we have the existence of a limit flow (follow the proof of Proposition 3.2.10).
The claimed properties of such limit flow are immediate by the above discussion and by the fact that any compact hypersurface cannot give rise to an eternal flow by Corollary 2.2.5. The only point requiring a justification is the embeddedness, if the initial hypersurface is embedded.
In this case, by Proposition 2.2.7, all the hypersurfaces in the flows φ_k are embedded at every time, then the only possibility for M^∞_s not to be embedded is if two or more of its regions "touch" each other at some point $y \in \mathbb{R}^{n+1}$ with a common tangent hyperplane.
We define the monotone nondecreasing function $G(t) = \max_{\substack{s\in[0,t]\\ p\in M}} |\mathrm{A}(p,s)|$ and we choose a smooth, monotone nondecreasing function $K : [0,T) \to \mathbb{R}^+$ such that $G(t) \leq K(t) \leq 2G(t)$ for every $t \in [0,T)$.
Then, we consider the following open set $\Omega_\varepsilon \subset M \times M \times [0,T)$ given by $\{(p,q,t) \,|\, d_{g(t)}(p,q) \leq \varepsilon/K(t)\}$, where $d_{g(t)}$ is the geodesic distance in the Riemannian manifold $(M, g(t))$. Let

$$B_\varepsilon = \inf_{\partial\Omega_\varepsilon} |\varphi(p,t) - \varphi(q,t)|K(t)$$

and suppose that $B_\varepsilon = 0$ for some $\varepsilon > 0$. This means that there exists a sequence of times $t_i \nearrow T$ and points p_i, q_i with $d_{g(t_i)}(p_i,q_i) = \varepsilon/K(t_i)$ and $|\varphi(p_i,t_i) - \varphi(q_i,t_i)|K(t_i) \to 0$, that is, $|\widetilde{\varphi}_i(p_i) - \widetilde{\varphi}_i(q_i)| \to 0$ and $d_{\widetilde{g}(s_i)}(p_i,q_i) = \varepsilon$, where $\widetilde{\varphi}_i$ is the rescaling of the hypersurface φ_{t_i} around the point $\varphi(p_i,t_i)$ by the dilation factor $K(t_i) \geq G(t_i)$.
As the curvatures A_i of these rescaled hypersurfaces $\widetilde{\varphi}_i$ satisfy

$$|\mathrm{A}_i(p)| = |\mathrm{A}(p,t_i)|/K(t_i) \leq |\mathrm{A}(p,t_i)|/G(t_i) \leq 1\,,$$

reasoning like in the proof of Proposition 3.2.10, we have a contradiction if ε >0 is small enough.
Now, fixing $\varepsilon > 0$ such that the above constant B_ε is positive and looking at the function

$$L(p,q,t) = |\varphi(p,t) - \varphi(q,t)|K(t)$$

on $\complement\Omega_\varepsilon \subset M \times M \times [0,T)$, we have that if the minimum of L at any time t (which is positive as the hypersurfaces are embedded) is lower than B_ε, then such minimum is not taken on the boundary of the set but in its interior, say at a pair (p,q). Then, we compute at the point (p,q,t)

$$\begin{aligned}\frac{\partial L(p,q,t)}{\partial t} &= K(t)\frac{\partial}{\partial t}|\varphi(p,t) - \varphi(q,t)| + |\varphi(p,t) - \varphi(q,t)|K'(t)\\ &\geq K(t)\frac{\partial}{\partial t}|\varphi(p,t) - \varphi(q,t)|\end{aligned}$$

and a geometric argument analogous to the one in the proof of Proposition 2.2.7 shows that this last partial derivative is nonnegative (when it exists, almost everywhere). Then, by means of the maximum principle (Hamilton's trick, Lemma 2.1.3) we conclude that when the minimum of L at time t is lower than B_ε it is nondecreasing.
Hence, there is a positive lower bound C_ε on

$$\inf_{\complement\Omega_\varepsilon} |\varphi(p,t) - \varphi(q,t)|K(t)\,,$$

consequently,

$$\inf_{\complement\Omega_\varepsilon} |\varphi(p,t) - \varphi(q,t)|G(t) \geq C_\varepsilon/2 > 0\,.$$

Now notice that, for the pairs (p_k,t_k) coming from formula (4.1.1) we have $|\mathrm{A}(p_k,t_k)| = G(t_k)$, otherwise there would exist a time $t < t_k$ with $\max_{p\in M} |\mathrm{A}(p,t)| > |\mathrm{A}(p_k,t_k)|$ which is in contradiction with the maximum in the right hand side of equation (4.1.1).

As $|\mathrm{A}(p_k,t_k)| \geq G(t)$ for every $t \leq t_k$, fixed $\omega,\delta > 0$, by inequality (4.1.2) we have definitely $\max_{p\in M} |\mathrm{A}_k(p,s)| \leq (1+\delta)$ for every $s \leq \omega$, hence

$$\begin{aligned} G(s/|\mathrm{A}(p_k,t_k)|^2 + t_k) &= \max_{\substack{r\leq s\\ p\in M}} |\mathrm{A}(p, r/|\mathrm{A}(p_k,t_k)|^2 + t_k)| \\ &\leq \max_{\substack{r\leq \omega\\ p\in M}} |\mathrm{A}_k(p,r)||\mathrm{A}(p_k,t_k)| \\ &\leq (1+\delta)|\mathrm{A}(p_k,t_k)|\,. \end{aligned} \tag{4.1.3}$$

If $s \in [0,\omega]$ and $d_{g_k(s)}(p,q) > \varepsilon$, definitely

$$\begin{aligned} d_{g(s/|\mathrm{A}(p_k,t_k)|^2+t_k)}(p,q) &= d_{g_k(s)}(p,q)/|\mathrm{A}(p_k,t_k)| \\ &= d_{g_k(s)}(p,q)/G(t_k) \\ &\geq \frac{\varepsilon}{G(s/|\mathrm{A}(p_k,t_k)|^2+t_k)} \\ &\geq \frac{\varepsilon}{K(s/|\mathrm{A}(p_k,t_k)|^2+t_k)} \end{aligned}$$

hence, $(p,q,s/|\mathrm{A}(p_k,t_k)|^2 + t_k) \in \complement\Omega_\varepsilon$.
If instead $s \leq 0$, we define $L(s) = \sup_{M_s^\infty} |\mathrm{A}_\infty| \leq 1$ and we see that if $L(s) = 0$ for some $s \leq 0$ then M_s^∞ is a hyperplane and the limit flow is flat till $s = 0$ (by uniqueness of the flow as A_∞ is bounded, see Remark 1.5.4), which is impossible as $|\mathrm{A}_\infty(0,0)| = 1$, hence $L(s) > 0$. Then, for every $s \leq 0$ we must have definitely

$$G(s/|\mathrm{A}(p_k,t_k)|^2 + t_k)/|\mathrm{A}(p_k,t_k)| \geq L(s)/2$$

and if $d_{g_k(s)}(p,q) > 2\varepsilon/L(s)$,

$$\begin{aligned} d_{g(s/|\mathrm{A}(p_k,t_k)|^2+t_k)}(p,q) &= d_{g_k(s)}(p,q)/|\mathrm{A}(p_k,t_k)| \\ &> \frac{2\varepsilon}{|\mathrm{A}(p_k,t_k)|L(s)} \\ &\geq \frac{\varepsilon}{G(s/|\mathrm{A}(p_k,t_k)|^2+t_k)} \\ &\geq \frac{\varepsilon}{K(s/|\mathrm{A}(p_k,t_k)|^2+t_k)} \end{aligned}$$

hence, also in this case $(p,q,s/|\mathrm{A}(p_k,t_k)|^2 + t_k) \in \complement\Omega_\varepsilon$.
Then in both cases, if $d_{g_k(s)}(p,q) > \min\{\varepsilon, 2\varepsilon/L(s)\} = \varepsilon > 0$ (notice that $\varepsilon < 2\varepsilon/L(s)$ as $L(s) \leq 1$),

$$\Big|\varphi(p, s/|\mathrm{A}(p_k,t_k)|^2 + t_k) - \varphi(q, s/|\mathrm{A}(p_k,t_k)|^2 + t_k)\Big|\, G(s/|\mathrm{A}(p_k,t_k)|^2 + t_k) \geq C_\varepsilon/2 > 0$$

and by inequality (4.1.3) it follows that definitely

$$\begin{aligned} |\varphi_k(p,s) - \varphi_k(q,s)| &= |\varphi(p, s/|\mathrm{A}(p_k,t_k)|^2 + t_k) - \varphi(q, s/|\mathrm{A}(p_k,t_k)|^2 + t_k)|\,|\mathrm{A}(p_k,t_k)| \\ &\geq \frac{C_\varepsilon|\mathrm{A}(p_k,t_k)|}{2G(s/|\mathrm{A}(p_k,t_k)|^2+t_k)} \\ &\geq \frac{C_\varepsilon}{2(1+\delta)}\,. \end{aligned}$$

As ω and δ were arbitrary and the convergence is smooth, this conclusion passes to all the limit hypersurfaces M_s^∞, for every $s \in \mathbb{R}$. That is, if a couple of points of M_s^∞ has intrinsic distance larger than $\varepsilon > 0$, their extrinsic distance is bounded from below by some uniform positive constant. If $\varepsilon > 0$ is then chosen small enough such that any hypersurface with $|\mathrm{A}| \leq 1$ (like every M_s^∞) is an embedding when it is restricted to any intrinsic ball of radius smaller than ε, we are done. The hypersurfaces M_s^∞ cannot have self–intersections for every $s \in \mathbb{R}$, hence they are all embedded. □

Exercise 4.1.5. This blow up procedure can be applied also at a type I singularity. There are some differences and the sequence t_k must be chosen in order that $t_k \to T$, since it is not a consequence of the construction.
The limit mean curvature flow that one obtains is no more eternal but only *ancient*, that is, defined on some interval $(-\infty, \Omega)$ with $\Omega > 0$, and $|\mathrm{A}_\infty| \leq 1$ holds only on $(\infty, 0]$.
It is an open problem if this limit flow is actually homothetically shrinking, in general.

Remark 4.1.6. Differently by the case of a type I singularity of the flow, we did not and we are not going to define any concept of singular point here. Moreover, it is quite conceivable and actually possible that while we are dealing with a type II singularity via the above Hamilton's procedure, in some other zone of the hypersurface the curvature is locally blowing up at the rate of a type I singularity or even mild singular points could be present, see Remark **??**. These latter, anyway, in the case of an embedded evolving hypersurface, can be excluded by the same argument of Remark **??**, based on White's Theorem 3.2.21 (which holds in general without any bound on the blow up rate of the curvature). In this situation too, it is unknown to the author if the presence of such points can be excluded also for general hypersurfaces or at least in the case of nonnegative mean curvature.

The analysis of singularities in the type II case is then reduced to classify these eternal flows with bounded curvature (and its covariant derivatives) with the extra property that the norm of the second fundamental form takes its maximum, equal to one, at some point in space and time.

Examples of this class are the *translating* mean curvature flows (with bounded second fundamental form and $|\mathrm{A}|$ achieving its maximum), that is, hypersurfaces $M \subset \mathbb{R}^{n+1}$ such that during the motion do not change their shape but simply move in a fixed direction with constant velocity. We have seen in Proposition 1.4.2 that this condition is equivalent to the existence of a vector $v \in \mathbb{R}^{n+1}$ such that $\mathrm{H}(p) = \langle v \,|\, \nu(p)\rangle$ at every point $p \in M$. Clearly, by comparison with spheres, these hypersurfaces cannot be compact.

Open Problem 4.1.7. Classify all the eternal mean curvature flows of complete, connected, hypersurfaces in $\mathbb{R}^{n+1}$ such that A and its covariant derivatives are uniformly bounded and $|\mathrm{A}|$ takes its maximum at some point in space–time. The same problem assuming embeddedness or supposing that the flow comes from Hamilton's blow up procedure.

Another problem is the analogous classification for *ancient* complete flows with bounded curvature at every fixed time (see the discussion in [121, page 536]). For closed convex curves, this problem has been solved by Daskalopoulos, Hamilton and Sesum [31]. The higher dimensional case was recently studied by Brendle, Huisken and Sinestrari.
Finally, the same questions can be asked also for the *immortal* flows, that is, defined on $[0, +\infty)$.

In view of the results of the next section, we also state the following.

Open Problem 4.1.8. All the eternal mean curvature flows of complete hypersurfaces in $\mathbb{R}^{n+1}$ coming from Hamilton's blow up procedure are translating flows? At least if they are embedded?

These problems are difficult in general, but like in the type I singularity case, if the evolving hypersurfaces are mean convex ($\mathrm{H} \geq 0$) or if we are dealing with curves in the plane, they have a positive answer. This will be the subject of the next sections.

We underline that the "bad blow up rate" is an obstacle to the use of Huisken's monotonicity formula in the context of type II singularities, an exception will be discussed in Section **??**.

We conclude this section by giving Hamilton's line of proof of Theorem 3.3.8, which is different from the original one.

Proof of Theorem 3.3.8. Let T be the maximal time of smooth existence of the mean curvature flow of an n–dimensional convex hypersurface. By the results of Section 2.5, in particular Proposition 2.5.8, we have that after any positive time $\mathrm{H} > 0$ and there exists a positive constant α, independent of time, such that $\mathrm{A} \geq \alpha \mathrm{H} g$ as forms.
If at time T we have a type II singularity, we get an unbounded, eternal convex blow up limit flow with $\mathrm{H} \geq 0$, using Hamilton's procedure. By the strong maximum principle, actually $\mathrm{H} > 0$

for every time (otherwise $\mathrm{H} \equiv 0$ everywhere, but this and the convexity would imply that the limit flow is simply a fixed hyperplane) and the condition $\mathrm{A} \geq \alpha \mathrm{H} g$ passes to the limit. Then, by the following theorem of Hamilton [60], all the hypersurfaces of the limit flow are compact, in contradiction with the unboundedness, hence type II singularities cannot develop.

Theorem 4.1.9. *Let M be a smooth, complete, strictly convex, n–dimensional hypersurface in the Euclidean space, with $n \geq 2$. Suppose that for some $\alpha > 0$ its second fundamental form is α–pinched in the sense that $\mathrm{A} \geq \alpha \mathrm{H} g$, where g is the induced metric and H its mean curvature. Then M is compact.*

Dealing with type I singularities, any blow up limit is embedded, strictly convex and compact, again by this theorem. Hence, by Theorem 3.3.5 it can be only the sphere $\mathbb{S}^n(\sqrt{n})$.
This implies that the full sequence of rescaled hypersurfaces converges in C^∞ to such sphere.
Finally, as the blow up limit is unique and compact, the original hypersurface shrinks to a point in finite time. □

4.2 Hypersurfaces with Nonnegative Mean Curvature

We shall now consider the formation of type II singularities for hypersurfaces which are mean convex, that is, with nonnegative mean curvature everywhere.
An important result for the analysis of singularities of mean convex hypersurfaces is the following estimate on the elementary symmetric polynomials of the curvatures S_k proved in [73], which holds in general for any mean curvature flow.

Theorem 4.2.1 (Huisken–Sinestrari [73])**.** *Let $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact, mean convex, immersed hypersurface, for $n \geq 2$. Then, for any $\eta > 0$ there exists a positive constant $C = C(\eta, \varphi_0)$ such that $\mathrm{S}_k \geq -\eta \mathrm{H}^k - C$ for any $k = 2, \dots, n$ at every point of M and $t \in [0,T)$.*

This estimate easily implies the following one, which has a more immediate interpretation.

Corollary 4.2.2. *Under the same hypotheses of the previous theorem, for any $\eta > 0$ there exists a positive constant $C = C(\eta, \varphi_0)$ such that $\lambda^{\min} \geq -\eta \mathrm{H} - C$ at every point of M and $t \in [0,T)$, where $\lambda^{\min}$ is the smallest eigenvalue of the second fundamental form.*

The interest in the above estimates lies in the fact that η can be chosen arbitrarily small and C is a constant not depending on the curvatures and time. Thus, roughly speaking, we see that the negative curvatures become negligible with respect to the others when the singular time is reached, as H is going to $+\infty$. This implies that the second fundamental form of the hypersurface becomes asymptotically nonnegative definite at a singularity.
Let us observe that these results cannot be valid for general hypersurfaces, even in low dimension. Indeed, Angenent's homothetically shrinking torus in [17] has a behavior which is incompatible with these convexity estimates.

Proposition 4.2.3. *If $n \geq 2$ and the initial hypersurface is mean convex, the limit flow M_s^∞ obtained by the Hamilton's procedure described in the previous section, consists of convex hypersurfaces.*

Proof. First, since we are taking the limit of hypersurfaces with $\mathrm{H} \geq 0$ the limit also is mean convex. By the strong maximum principle applied to the evolution equation for the mean curvature of the limit flow $\partial_t \mathrm{H}_\infty = \Delta \mathrm{H}_\infty + \mathrm{H}_\infty |\mathrm{A}_\infty|^2$, we actually have $\mathrm{H}_\infty(p,t) > 0$ for every point in space and time, otherwise H_∞ is identically zero and also A_∞ would be identically zero (by Proposition 2.4.1 and the pinching estimates in Corollary 2.4.3, which are invariant by rescaling and pass to the limit), in contradiction with the fact that the limit flow is nonflat.
Fixing any $\eta > 0$ and a pair (p,s) with $p \in M_s^\infty$, if $Q_k \to +\infty$ is the rescaling factor for the flow φ_k and $q_k \in M$ is such that $p_k = \varphi_k(q_k, s)$ converges to p as $k \to \infty$, we have $\mathrm{H}_k(q_k, s) = \mathrm{H}(q_k, s/Q_k^2 + t_k)/Q_k \to \mathrm{H}_\infty(p,t) > 0$ hence $\mathrm{H}(q_k, s/Q_k^2 + t_k) \to +\infty$. Now, since by Corollary 4.2.2 there exists a constant $C > 0$ such that $\lambda^{\min} \geq -\eta \mathrm{H} - C$ for the original flow φ and $\mathrm{H} > \varepsilon$ at least

for every $t > \delta > 0$, we have $\lambda^{\min}/\mathrm{H} \geq -\eta - C/\mathrm{H}$ everywhere. When we rescale the hypersurfaces we get

$$\frac{\lambda_k^{\min}(q_k, s)}{\mathrm{H}_k(q_k, s)} = \frac{\lambda^{\min}(q_k, s/Q_k^2 + t_k)}{\mathrm{H}(q_k, s/Q_k^2 + t_k)} \geq -\eta - \frac{C}{\mathrm{H}(q_k, s/Q_k^2 + t_k)}$$

and sending $k \to \infty$ we conclude $\lambda_\infty^{\min}(p, s)/\mathrm{H}_\infty(p, s) \geq -\eta$.
Since $\eta > 0$ was arbitrary and this argument holds for every pair (p, s) with $p \in M_s^\infty$, the second fundamental form is nonnegative definite on the whole limit flow, hence all the hypersurfaces are convex. □

Remark 4.2.4. Instead of using Corollary 4.2.2, one can apply the same argument directly to the estimates of Theorem 4.2.1 obtaining that all the elementary symmetric polynomials of the eigenvalues of the second fundamental form are nonnegative at every point in space and time for the limit flow. By relations (2.5.1) the conclusion follows.

Remark 4.2.5. This result also holds if the Hamilton's procedure is applied type I singularities (see Exercise 4.1.5).

Remark 4.2.6. This proposition (in a slightly stronger form) has also been obtained by White [122] by completely different techniques. His approach also works for the subsequent singularities of "weak" mean curvature flows which continue after the first singular time.

The hypersurfaces of the limit flow are convex, but in general not strictly convex. However, if they are not strictly convex then they necessarily split as the product of a flat factor with a strictly convex one, as shown by the following result.

Proposition 4.2.7 (Theorem 4.1 in [73]). *If any of the convex hypersurfaces of the limit flow M_s^∞ is not strictly convex, then (up to a rigid motion) $M_s^\infty = N_s^m \times \mathbb{R}^{n-m}$, where $1 \leq m < n$ and N_s^m is a family of strictly convex, m–dimensional, complete hypersurfaces moving by mean curvature in $\mathbb{R}^{m+1}$.*

Proof. The proof is based on Hamilton's strong maximum principle for tensors in [56, Section 8], Theorem B.1.3), which holds also if the manifold is not compact (as it is in our case).
If $m(s) \in \mathbb{N}$ is the minimal rank of A_∞ on M_s^∞, arguing as in Remark 2.5.6 this integer valued function is nondecreasing. Letting $m < n$ be its global minimum which is realized at some point of $M_{s_0}^\infty$, it follows that $m(s) = m$ for every $s \leq s_0$. Again by the argument in Remark 2.5.6, for every $s \leq s_0$ the hypersurface M_s^∞ must contain an $(n-m)$–dimensional affine subspace of $\mathbb{R}^{n+1}$ which is invariant under parallel transport and in time. Clearly, such subspace is the same for all $s \leq s_0$.
Thus, the limit flow for $s \in (-\infty, s_0]$ splits as a product of an $(n-m)$–dimensional flat part and a family of strictly convex m–dimensional hypersurfaces $N_s^m \subset \mathbb{R}^{m+1}$ evolving by mean curvature. By uniqueness of the flow as A_∞ is bounded (see the discussion in Remark 1.5.4), this must hold also for every $s > s_0$. □

Exercise 4.2.8. For a type I singularity of the mean curvature flow of a mean convex, *embedded* initial hypersurface the Hamilton's procedure (see Exercise 4.1.5) gives a flow M_s^∞ which is of the form $\mathbb{S}_s^m \times \mathbb{R}^{n-m}$, for some $1 \leq m \leq n$ where $\mathbb{S}_s^m$ is an m–dimensional shrinking sphere.

In the case of the evolution of mean convex hypersurfaces in a time interval $[0, T)$, by Proposition 2.4.2 and Corollary 2.4.3, the mean curvature H and $|\mathrm{A}|$ are comparable quantities, that is, there exists a constant α, independent of time such that $\alpha|\mathrm{A}| \leq \mathrm{H} \leq \sqrt{n}|\mathrm{A}|$ for $t \in [\delta, T)$. This implies that we can modify Hamilton's blow up procedure, substituting H^2 in place of $|\mathrm{A}|^2$ in equation (4.1.1), with the same estimates on the second fundamental form and its covariant derivatives.
We then still get an eternal smooth limit flow, complete with bounded curvature and its covariant derivatives, with the only difference that this time it is the mean curvature H which gets a global maximum equal to one at time zero. This will be crucial to continue the analysis in the next sections.
Analogously, it is easy to see that the conclusions of Propositions 4.2.3 and 4.2.7 are not affected

by this modification so also in this case the limit flow consists of convex hypersurfaces.
We call this limit flow Hamilton's *modified* blow up limit.

Remark 4.2.9. Notice that for curves in $\mathbb{R}^2$ the two procedures coincide as $|\mathrm{A}| = |\mathrm{H}| = |k|$, where k is the usual curvature of a curve in the plane.

As the argument leading to Proposition 4.2.3 does not work in the one–dimensional case of curves, we deal with this latter separately in the next section.

4.3 The Special Case of Curves

Again, the case of a closed curve in $\mathbb{R}^2$ is special.
We suppose to deal with a generic initial closed curve, smoothly immersed in the plane $\mathbb{R}^2$ and moving by mean curvature $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ where at time T we have a type II singularity. Setting ξ and k to be respectively the arclength and the curvature of γ_t, we have the evolution equation $\partial_t k = k_{\xi\xi} + k^3$, then we define the function $z(t) = \#\{p \in \gamma_t \,|\, k(p) = 0\}$ "counting" the number of points on γ_t such that $k = 0$.
We need the following result of Angenent in [16, Proposition 1.2] and [15, Section 2] (see [13] for the proof).

Proposition 4.3.1. *If we have a mean curvature flow of a (possibly unbounded) curve in $\mathbb{R}^2$ which is not a line, in an open interval of time, at every fixed time the points where k is zero are isolated in space. In particular, this implies that for a closed curve, the function z is finite at every time.*
The function z is nonincreasing during the flow, hence if at some time it is finite, it remains finite.
Finally, if at some point $p \in \gamma_t$ we have $k(p) = 0$ and $k_\xi(p) = 0$ then the zero point p for k immediately vanishes. To be precise, this means that there exists a small space interval I around p and a small $r > t$ such that k is never zero in $I \times (t, r)$.

We only mention that the proof is based on the application of the maximum principle to the above evolution equation for the curvature.

By this proposition, in our case we can define $\mathcal{I}_t$ to be the finite family of open intervals on γ_t where $k \neq 0$ and the following computation is justified,

$$\begin{aligned} \frac{d}{dt}\int_{\gamma_t} |k|\,d\xi &= \sum_{I\in\mathcal{I}_t}\int_I \Big[(\mathrm{sign}k)(k_{\xi\xi}+k^3) - |k|^3\Big]\,d\xi \\ &= \sum_{I\in\mathcal{I}_t}\int_I (\mathrm{sign}k)k_{\xi\xi}\,d\xi \\ &= -\,2\sum_{\{p\in\gamma_t\,|\,k(p)=0\}} |k_\xi(p)|\,. \end{aligned}$$

Hence, the integral $\int_{\gamma_t}|k|\,d\xi$, which is positive and finite at every time by compactness, is not increasing during the flow so it converges to some value $L \geq 0$ as $t \to T$, moreover it is scaling invariant.
Then we have, for every $t_1 < t_2$,

$$\int_{\gamma_{t_1}} |k|\,d\xi - \int_{\gamma_{t_2}} |k|\,d\xi = 2\int_{t_1}^{t_2} \sum_{\{p\in\gamma_t\,|\,k(p)=0\}} |k_\xi(p)|\,dt\,.$$

If now we apply the Hamilton's procedure, calling γ^n_s the rescaled curves at step n with curvatures k_n and denoting by $K_n \to +\infty$ the rescaling factor, we have for every interval $(a, b) \subset \mathbb{R}$

$$2\int_a^b \sum_{\{p\in\gamma_s^n \,|\, k_n(p)=0\}} |\partial_\xi k_n|\, ds = \int_{\gamma_a^n} |k_n|\, d\xi - \int_{\gamma_b^n} |k_n|\, d\xi \tag{4.3.1}$$
$$= \int_{\gamma_{\frac{a}{K_n}+t_n}} |k|\, d\xi - \int_{\gamma_{\frac{b}{K_n}+t_n}} |k|\, d\xi\,,$$

since $\int_{\gamma_t} |k|\, d\xi$ is scaling invariant and where, by simplicity, we used ξ also for the arclength of the rescaled curves.

It is easy to see that the integral $\int_a^b \sum_{\{p\in\gamma_s \,|\, k(p)=0\}} |k_\xi|\, ds$ is lower semicontinuous under the smooth local convergence of curves, hence

$$\begin{aligned}\int_a^b \sum_{\{p\in\gamma_s^\infty \,|\, k_\infty(p)=0\}} |\partial_\xi k_\infty|\, ds &\leq \lim_{n\to\infty} \int_a^b \sum_{\{p\in\gamma_s^n \,|\, k_n(p)=0\}} |\partial_\xi k_n|\, ds\\ &= \lim_{n\to\infty} \Big(\int_{\gamma_{\frac{a}{K_n}+t_n}} |k|\, d\xi - \int_{\gamma_{\frac{b}{K_n}+t_n}} |k|\, d\xi \Big)\\ &= 0\end{aligned}$$

for the limit flow γ_s^∞, as both $\frac{a}{K_n} + t_n$ and $\frac{b}{K_n} + t_n$ converge to T, hence both integrals in equation (4.3.1) converge to L. As a and b were arbitrary, we conclude for almost every $s \in \mathbb{R}$

$$\sum_{\{p\in\gamma_s^\infty \,|\, k_\infty(p)=0\}} |\partial_\xi k_\infty(p)| = 0\,,$$

that is, $\partial_\xi k_\infty$ is zero at every point in space and time where k_∞ is zero.
Again by means of Proposition 4.3.1, fixing $s \in \mathbb{R}$ and choosing any small $r > s$, the zero points of the curvature vanish for the curve γ_r^∞, hence $k_\infty > 0$ on γ_r^∞ for every $r > s$, as it is a condition which is preserved under the flow. Since we can draw this conclusion for almost every $s \in \mathbb{R}$, at every time the flow γ_s^∞ consists of curves such that k_∞ is never zero.
Hence, we have the following one–dimensional analogue of Proposition 4.2.3.

Proposition 4.3.2. *The limit flow γ_s^∞ obtained by the Hamilton's procedure at a type II singularity of the evolution by curvature of any initial closed curve, consists of curves such that k_∞ is never zero, in particular if the initial curve was embedded all such curves are strictly convex.*

Remark 4.3.3. We underline that we did not assume that the initial curve was embedded. The above conclusion holds for the flow of any immersed closed curve in the plane (like the results of the previous section holding for general immersed–only hypersurfaces).

4.4 Hamilton's Harnack Estimate for Mean Curvature Flow

We have seen in the previous two sections that if a closed curve or a compact hypersurface with $\mathrm{H} \geq 0$ develops a type II singularity then the limit of the rescaled flows by the "modified" Hamilton's procedure is an eternal mean curvature flow of convex, complete, hypersurfaces such that H takes its maximum in space and time at some point. We want now to see that this implies that such limit flow is translating, this is obtained by means of the following two deep results of Hamilton in [62].

Theorem 4.4.1 (Harnack Estimate for Mean Curvature Flow)**.** *Let $\varphi : M \times (T_0, T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a complete, convex hypersurface with bounded second fundamental form at every*

time.
Let X be a time dependent, smooth tangent vector field on M. Then the following inequality holds,

$$\frac{\partial \mathrm{H}}{\partial t} + \frac{\mathrm{H}}{2(t-T_0)} + 2\langle \nabla \mathrm{H} \,|\, X\rangle + h_{ij}X^iX^j \geq 0$$

for every $t \in (T_0, T)$.

Theorem 4.4.2. *Let $\varphi : M \times (-\infty, T) \to \mathbb{R}^{n+1}$ be an ancient mean curvature flow of a complete, strictly convex hypersurface with bounded second fundamental form at every time and such that H takes its maximum in space and time. Then, φ is a translating flow with some constant velocity $v \in \mathbb{R}^{n+1}$, that is, it satisfies $\mathrm{H} = \langle v \,|\, \nu\rangle$ at every point in space and time.*

The proofs of these two theorems involve some smart and heavy computations with a strong use of the maximum principle, we show the complete proof of Theorem 4.4.1 only in the one–dimensional, compact case and of Theorem 4.4.2 only in the one–dimensional case, referring the reader to the original paper [62] (see also [57]).

Proof of Theorem 4.4.1 – One–Dimensional Compact Case. As the evolving curves are compact, the curvature k and all its derivatives are bounded in $(C, T-\varepsilon)$, for every $\varepsilon > 0$ and $C \in (T_0, T-\varepsilon)$. Moreover, by Proposition 2.4.1, in the same interval, $k > k_0 > 0$ for some positive constant k_0. Since any tangent vector field X can be written as $X = \lambda\tau$ for some function $\lambda : \mathbb{S}^1 \times (T_0, T) \to \mathbb{R}$, we define the Hamilton's quadratic

$$Z(\lambda) = \partial_t k + \frac{k}{2(t-C)} + 2\lambda k_s + k\lambda^2 = k_{ss} + k^3 + \frac{k}{2(t-C)} + 2\lambda k_s + k\lambda^2$$

which is clearly bounded from below by

$$Z = k_{ss} + k^3 - k_s^2/k + \frac{k}{2(t-C)}\,.$$

We also define

$$W = k_{ss} + k^3 - k_s^2/k$$

and we start computing the evolution equation for this latter quantity by means of the evolution equations in Remark 2.3.2,

$$\begin{aligned}
(\partial_t - \partial_{ss})W =\,& \partial_t k_{ss} - \frac{2k_s\partial_t k_s}{k} + \frac{k_s^2 k_t}{k^2} + 3k^2k_t - k_{ssss} + \frac{2k_sk_{sss}}{k} + \frac{2k_{ss}^2}{k}\\
& - \frac{5k_s^2k_{ss}}{k^2} + \frac{2k_s^4}{k^3} - 6kk_s^2 - 3k^2k_{ss}\\
=\,& \partial_s\partial_t k_s + k^2k_{ss} - \frac{2k_s\partial_s k_t}{k} - 2kk_s^2 + \frac{k_s^2k_{ss}}{k^2} + kk_s^2 + 3k^2k_{ss} + 3k^5\\
& - k_{ssss} + \frac{2k_sk_{sss}}{k} + \frac{2k_{ss}^2}{k} - \frac{5k_s^2k_{ss}}{k^2} + \frac{2k_s^4}{k^3} - 6kk_s^2 - 3k^2k_{ss}\\
=\,& \partial_{ss}(k_{ss} + k^3) + 2k^2k_{ss} - 5kk_s^2 - \frac{2k_s\partial_s(k_{ss}+k^3)}{k}\\
& + \frac{k_s^2k_{ss}}{k^2} + 3k^5 - k_{ssss} + \frac{2k_sk_{sss}}{k} + \frac{2k_{ss}^2}{k} - \frac{5k_s^2k_{ss}}{k^2} + \frac{2k_s^4}{k^3}\\
=\,& k_{ssss} + 5k^2k_{ss} - 5kk_s^2 - \frac{2k_sk_{sss}}{k}\\
& - \frac{4k_s^2k_{ss}}{k^2} + 3k^5 - k_{ssss} + \frac{2k_sk_{sss}}{k} + \frac{2k_{ss}^2}{k} + \frac{2k_s^4}{k^3}\\
=\,& -5kk_s^2 + 3k^5 + \frac{2k_s^4}{k^3} + 5k^2k_{ss} + \frac{2k_{ss}^2}{k} - \frac{4k_s^2k_{ss}}{k^2}\,.
\end{aligned}$$

As $k_{ss} = (W + k_s^2/k - k^3)$, substituting we get

$$
\begin{aligned}
(\partial_t - \partial_{ss})W = & -5kk_s^2 + 3k^5 + \frac{2k_s^4}{k^3} \\
& + 5k^2(W + k_s^2/k - k^3) \\
& + \frac{2(W + k_s^2/k - k^3)^2}{k} - \frac{4k_s^2(W + k_s^2/k - k^3)}{k^2} \\
= & -5kk_s^2 + 3k^5 + \frac{2k_s^4}{k^3} \\
& + 5k^2W + 5kk_s^2 - 5k^5 \\
& + \frac{2W^2}{k} + \frac{2k_s^4}{k^3} + 2k^5 + \frac{4Wk_s^2}{k^2} - 4Wk^2 - 4kk_s^2 \\
& - \frac{4k_s^2(W + k_s^2/k - k^3)}{k^2} \\
= & \frac{2W^2}{k} + Wk^2 \,.
\end{aligned}
\tag{4.4.1}
$$

We notice that, since $k > k_0 > 0$, by the maximum principle if W is positive at some time it remains positive.
As $Z = W + k/(2(t - C))$, we then get

$$
\begin{aligned}
(\partial_t - \partial_{ss})Z = & (\partial_t - \partial_{ss})W + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2} \\
= & \frac{2W^2}{k} + Wk^2 + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2} \\
= & \frac{2(Z - k/(2(t-C)))^2 + k^3(Z - k/(2(t-C)))}{k} + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2} \\
= & \frac{2Z^2 + k^2/(2(t-C)^2) - 2Zk/(t-C)}{k} + \frac{k^3Z - k^4/(2(t-C))}{k} \\
& + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2} \\
= & \frac{2Z^2}{k} - \frac{2Z}{t-C} + k^2Z \,.
\end{aligned}
$$

As $k > k_0 > 0$ the term $k/(2(t - C))$ diverges as $t \to C^+$ and W is bounded from below, then we have that Z goes uniformly to $+\infty$ as $t \to C^+$. Hence, Z is positive in $\mathbb{S}^1 \times (C, C + \delta)$ for some $\delta > 0$ and by the maximum principle it cannot get zero on γ_t for every $t \in (C, T - \varepsilon)$.
As $Z(\lambda) \geq Z > 0$ for every function $\lambda : M \times (T_0, T) \to \mathbb{R}$, sending $\varepsilon \to 0$ and $C \to T_0$ we have the thesis of the theorem. □

Remark 4.4.3. When the curves γ_t are not compact there are two nontrivial technical points to take care of: the possible nonexistence of the minimum in space of $Z(\,\cdot\,, t)$ and the fact that it is not granted that $\lim_{t\to C^+} \inf_{\gamma_t} Z(\,\cdot\,, t) = +\infty$, as k could go to zero at infinity (possibly, a value $k_0 > 0$ such that $k > k_0$ uniformly does not exist). This requires a perturbation of Z in space with a function growing enough at infinity and the addition to Z of another function assuring that the resulting term uniformly diverges as $t \to C^+$ (see [57, 62] for the details).

Remark 4.4.4. The higher complexity of the proof in dimension larger than one is essentially due to the fact that the minimum of the quadratic

$$
Z(X) = \frac{\partial \mathrm{H}}{\partial t} + \frac{\mathrm{H}}{2(t-C)} + 2\langle \nabla \mathrm{H} \,|\, X \rangle + h_{ij}X^iX^j \,,
$$

which is given by

$$Z = \frac{\partial \mathrm{H}}{\partial t} + \frac{\mathrm{H}}{2(t-C)} - (\mathrm{A}^{-1})^{pq}\nabla_p \mathrm{H} \nabla_q \mathrm{H}\,,$$

is clearly more difficult to deal with than in the one–dimensional case (here $(\mathrm{A}^{-1})^{pq}$ denotes the *inverse matrix* of the second fundamental form h_{ij}, that is $(\mathrm{A}^{-1})^{pq} h_{qr} = \delta^p_r$).
Anyway, after a quite long computation one can see that

$$\frac{\partial Z}{\partial t} - \Delta Z = 2g^{ij}(\mathrm{A}^{-1})^{kl} J_{ik} J_{jl} + \Big(|\mathrm{A}|^2 - \frac{2}{t-C}\Big) Z \geq \Big(|\mathrm{A}|^2 - \frac{2}{t-C}\Big) Z$$

where

$$J_{ik} = \nabla^2_{ik}\mathrm{H} + \mathrm{H} h^2_{ik} - (\mathrm{A}^{-1})^{pq}\nabla_p \mathrm{H} \nabla_q h_{ik} + \frac{h_{ik}}{2(t-C)}\,,$$

see [27, Chapter 15].
Actually, another possibility is to keep the vector field X generic and to compute the evolution equation for $Z(X)$, like in the original proof of Hamilton.

Proof of Theorem 4.4.2 – One–Dimensional Case. Suppose that we have an ancient curvature flow γ_t of complete, connected curves in the plane with $k > 0$. By Theorem 4.4.1 we have

$$Z = \partial_t k - k_s^2/k + k/(t - T_0) \geq 0$$

at every point and for every $t, T_0 \in \mathbb{R}$ with $T_0 < t < T$. Sending $T_0 \to -\infty$ we get

$$W = \partial_t k - k_s^2/k \geq 0\,.$$

As we computed in equation (4.4.1) that

$$(\partial_t - \partial_{ss})W = \frac{2W^2}{k} + Wk^2\,,$$

if W is zero at some point in space and time, it must be zero everywhere by the strong maximum principle. By hypothesis k takes a maximum at some point in space and time, hence at such point $k_t = k_s = 0$ which implies $W = 0$.
Thus, $k_t = k_s^2/k$ for all the curves of the flow, or equivalently $k_{ss} + k^3 - k_s^2/k = 0$.
If we set $v = -(k_s/k)\tau + k\nu$ as a vector field in $\mathbb{R}^2$ along γ_t, obviously $\langle v\,|\,\nu\rangle = k$, then

$$\begin{aligned}\partial_s v =& -(k_{ss}/k - k_s^2/k^2)\tau - (k_s/k)k\nu + k_s\nu - k^2\tau\\ =& -(-k^2 + k_s^2/k^2 - k_s^2/k^2)\tau - k^2\tau = 0\end{aligned}$$

and

$$\begin{aligned}\partial_t v =& (-k_{ts}/k + k_s k_t/k^2 - kk_s)\tau + (-k_s^2/k + k_t)\nu\\ =& (-k_{st}/k - kk_s + k_s^3/k^3 - kk_s)\tau\\ =& (-[\partial_s(k_s^2/k)]/k + k_s^3/k^3 - 2kk_s)\tau\\ =& (-2k_s k_{ss}/k^2 + 2k_s^2/k^3 - 2kk_s)\tau\\ =& -2\frac{k_s}{k}(k_{ss} - k_s^2/k + k^3)\tau = 0\,.\end{aligned}$$

Hence, as the curves of the flow are connected, v is a vector field along γ_t constant in space and time.
Since $k = \langle v\,|\,\nu\rangle$, we have that the curves γ_t move by translation under the mean curvature flow. □

Then, putting together Propositions 4.2.3, 4.2.7, 4.3.2 and Theorem 4.4.2, we have the following results.

Theorem 4.4.5. *The blow up limit flow obtained by the Hamilton's* modified *procedure at a type II singularity of the motion of a initial hypersurface with* $\mathrm{H} \geq 0$ *is a translating mean curvature flow of complete, nonflat, convex hypersurfaces with bounded curvature and its covariant derivatives, that is, it satisfies* $\mathrm{H} = \langle v \,|\, \nu \rangle$ *at every point in space and time.*
If any of the convex hypersurfaces of the limit flow is not strictly convex, then the limit flow splits as the product of an m*–dimensional strictly convex, translating flow as above and* $\mathbb{R}^{n-m}$*.*

Theorem 4.4.6. *The blow up limit flow obtained by the Hamilton's procedure at a type II singularity of the motion of a closed curve in the plane is a translating curvature flow of complete, nonflat curves with bounded curvature and its covariant derivatives. Moreover, for all the curves* $k > 0$*.*
Hence, this flow is given (up to rigid motions) by the grim reaper *(see Section 1.4).*

Remark 4.4.7. For curves in the plane, possibly with self–intersections, such that the initial curvature is never zero, this result was obtained via a different method by Angenent [15] (see also [3]), studying directly the parabolic equation satisfied by the curvature function.

In [122], White was able to exclude the possibility of getting as a blow up limit the product of a grim reaper with $\mathbb{R}^{n-1}$, when $n \geq 2$.

In dimension two, by this result of White and the analysis of Wang [119], the only possible blow up limit flow is given (up to a rigid motion) by the unique rotationally symmetric, translating hypersurface which is the graph of an entire, strictly convex function described by the ODE (1.4.1), in Section 1.4.

In general, without assuming the condition $\mathrm{H} \geq 0$, one could conjecture that blow up limits like the minimal catenoid surface M in $\mathbb{R}^3$ given by

$$\Omega = \big\{(x, y) \in \mathbb{R}^2 \times \mathbb{R} \,|\, \cosh |y| = |x|\big\}$$

cannot appear. See White [122], Ecker [36] for more details and the recent paper by Sheng and Wang [103].

4.5 Embedded Closed Curves in the Plane

In the special case of the evolution of an embedded closed curve in the plane, it is possible to exclude at all the type II singularities. This, together with the case of convex, compact, hypersurfaces (as we have seen in the proof of Theorem 3.3.8) are the only known cases in which this can be done in general.

By Theorem 4.4.6 and embeddedness, any blow up limit must a unit multiplicity grim reaper. We apply now a very geometric argument by Huisken in [70] in order to exclude also such possibility (see also [63] for another similar quantity).

Given the smooth flow γ_t of an initial embedded closed curve in some interval $[0, T)$, we know that the curve stays embedded during the flow so we can see every γ_t as a subset of $\mathbb{R}^2$. At every time $t \in [0, T)$, for every pair of points p and q in γ_t we define $d_t(p, q)$ to be the *geodesic* distance in γ_t of p and q, $|p - q|$ the standard distance in $\mathbb{R}^2$ and L_t the length of γ_t.
We consider the function $\Phi_t : \gamma_t \times \gamma_t \to \mathbb{R}$ defined as

$$\Phi_t(p, q) = \begin{cases} \frac{\pi|p-q|}{L_t} \big/ \sin \frac{\pi d_t(p,q)}{L_t} & \text{if } p \neq q, \\ 1 & \text{if } p = q\,, \end{cases}$$

which is a perturbation of the quotient between the extrinsic and the intrinsic distance of a pair of points on γ_t.
Since γ_t is smooth and embedded for every time, the function Φ_t is well defined and positive. Moreover, it is easy to check that even if d_t is not C^1 at the pairs of points such that $d_t(p, q) = L_t/2$, the function Φ_t is C^2 in the open set $\{p \neq q\} \subset \gamma_t \times \gamma_t$ and continuous on $\gamma_t \times \gamma_t$.
By compactness, the following minimum there exists,

$$E(t) = \min_{p,q \in \gamma_t} \Phi_t(p, q)\,.$$

We call this quantity *Huisken's embeddedness ratio*.
Since the evolution is smooth it is easy to see that the function $E : [0, T) \to \mathbb{R}$ is continuous.

Remark 4.5.1. The quantity E can be defined also for nonembedded closed curves, but in such case $E = 0$, indeed its positivity is equivalent to embeddedness.

Lemma 4.5.2 (Huisken [70]). *The function $E(t)$ is monotone increasing in every time interval where $E(t) < 1$.*

Proof. We start differentiating in time $\Phi_t(p, q)$,

$$\begin{aligned}
\frac{d}{dt}\Phi_t(p,q) &= \frac{\pi}{L_t}\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|}\Big/\sin\frac{\pi d_t(p,q)}{L_t} \\
&\quad+\left(\frac{\pi|p-q|}{L_t^2}\int_{\gamma_t}k^2\,ds\right)\Big/\sin\frac{\pi d_t(p,q)}{L_t} \\
&\quad-\frac{\pi^2|p-q|}{L_t^2}\cos\frac{\pi d_t(p,q)}{L_t}\left(\frac{d_t(p,q)}{L_t}\int_{\gamma_t}k^2\,ds-\int_q^p k^2\,ds\right)\Big/\sin^2\frac{\pi d_t(p,q)}{L_t} \\
&=\left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{1}{L_t}\int_{\gamma_t}k^2\,ds\right. \\
&\quad\left.-\frac{\pi}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\left(\frac{d_t(p,q)}{L_t}\int_{\gamma_t}k^2\,ds-\int_q^p k^2\,ds\right)\right]\Phi_t(p,q) \\
&=\left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{1}{L_t}\left(1-\frac{\pi d_t(p,q)}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\right)\int_{\gamma_t}k^2\,ds\right. \\
&\quad\left.+\frac{\pi}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\int_q^p k^2\,ds\right]\Phi_t(p,q)
\end{aligned}$$

where s is the arclength and k the curvature of γ_t. It is easy to see that being the function E the minimum of a family of uniformly locally Lipschitz functions, it is also locally Lipschitz, hence differentiable almost everywhere. Then, to prove the statement it is enough to show that $\frac{dE(t)}{dt} > 0$ for every time t such that this derivative exists. We will do that as usual, by Hamilton's trick (Lemma 2.1.3).
Let (p, q) be a minimizing pair at a differentiability time t and suppose that $E(t) < 1$. By the very definition of Φ_t, it must be $p \neq q$.
We set $\alpha = \pi d_t(p, q)/L_t$ and notice that $\alpha \cot \alpha < 1$ as $\alpha \in (0, \pi/2]$. Moreover, $\int_{\gamma_t} k^2\,ds \geq \left(\int_{\gamma_t} k\,ds\right)^2 / L_t \geq 4\pi^2/L_t$. Then, we have

$$\frac{d}{dt}E(t) \geq \left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha)+\frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\right]E(t)$$

that is,

$$\frac{d}{dt}\log E(t) \geq \frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha)+\frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\,, \qquad (4.5.1)$$

for any minimizing pair (p, q).
Assume that the curve is parametrized counterclockwise by arclength and that p and q are like in Figure 4.1.
We set $p(s) = \gamma_t(s_1 + s)$ with $p = \gamma_t(s_1)$, then, by minimality we have

$$0 = \frac{d}{ds}\Phi_t(p(s),q)\bigg|_{s=0} = \frac{\pi}{L_t}\frac{\langle p-q\,|\,\tau(p)\rangle}{|p-q|\sin\frac{\pi d_t(p,q)}{L_t}} - \frac{\pi|p-q|}{L_t\sin^2\frac{\pi d_t(p,q)}{L_t}}\cdot\frac{\pi\cos\frac{\pi d_t(p,q)}{L_t}}{L_t}$$

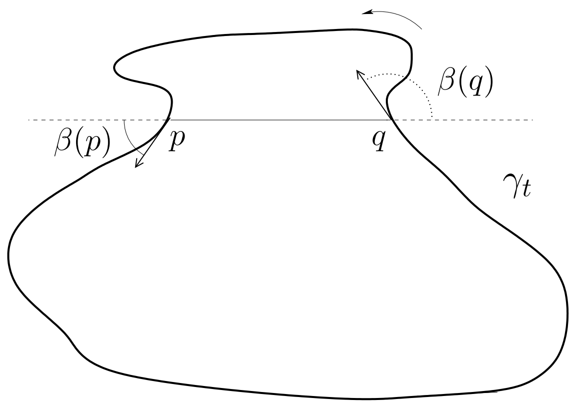

Figure 4.1:

where we denoted by $\tau(p)$ the oriented unit tangent vector to γ_t at p.
By this equality we get

$$\cos\beta(p) = \frac{\langle p-q\,|\,\tau(p)\rangle}{|p-q|} = \frac{\pi|p-q|}{L_t\sin\frac{\pi d_t(p,q)}{L_t}}\cos\frac{\pi d_t(p,q)}{L_t} = E(t)\cos\alpha$$

where $\beta(p)\in[0,\pi/2]$ is the angle between the vectors $p-q$ and $\tau(p)$.
Repeating this argument for the point q we get

$$\cos\beta(q) = -E(t)\cos\alpha$$

where, as before, $\beta(q)$ is the angle between $q-p$ and $\tau(q)$, see Figure 4.1. Clearly, it follows that $\beta(p)+\beta(q)=\pi$.
Notice that if one of the intersections of the segment $[p,q]$ with the curve is tangential, we would have $E(t)\cos\alpha = 1$ which is impossible as we assumed that $E(t) < 1$. Moreover, by the relation $\cos\beta(p) = E(t)\cos\alpha < \cos\alpha$ it follows that $\beta(p) > \alpha$.

We look now at the second variation of Φ_t at the same minimizing pair of points (p,q). With the same notation, if $p=\gamma_t(s_1)$ and $q=\gamma_t(s_2)$ we set $p(s)=\gamma_t(s_1+s)$ and $q(s)=\gamma_t(s_2-s)$. After a straightforward computation, one gets

$$\begin{aligned}
0 \leq\ & \frac{d^2}{ds^2}\Phi_t(p(s),q(s))\Big|_{s=0}\\
=\ & \frac{\pi}{L_t}\left(\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|} + \frac{4\pi^2|p-q|}{L_t^2}\right)\Big/\sin\frac{\pi d_t(p,q)}{L_t}\\
=\ & \left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2} + \frac{4\pi^2}{L_t^2}\right]E(t)\,.
\end{aligned}$$

Hence, getting back to inequality (4.5.1) we have

$$\begin{aligned}
\frac{d}{dt}\log E(t) \geq\ & \frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2} + \frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha) + \frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\\
\geq\ & -\frac{4\pi^2}{L_t^2}\alpha\cot\alpha + \frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\\
=\ & \frac{\pi\cot\alpha}{L_t}\left(\int_q^p k^2\,ds - \frac{4\pi}{L_t}\alpha\right),
\end{aligned}$$

so it remains to show that this last expression is positive. As

$$\int_p^q k^2\,ds \geq \left(\int_p^q k\,ds\right)^2 / d_t(p,q)$$

and noticing that $\int_p^q k\,ds$ is the angle between the tangent vectors $\tau(p)$ and $\tau(q)$ we have $\left(\int_p^q k\,ds\right)^2 = 4\beta(p)^2 < 4\alpha^2$, as we concluded before.
Thus,

$$\begin{aligned}\frac{d}{dt}\log E(t) &\geq \frac{\pi\cot\alpha}{L_t}\left(\int_q^p k^2\,ds - \frac{4\pi}{L_t}\alpha\right)\\ &> \frac{\pi\cot\alpha}{L_t}\left(\frac{4\alpha^2}{d_t(p,q)} - \frac{4\pi}{L_t}\alpha\right)\\ &= 0\end{aligned}$$

recalling that $\alpha = \pi d_t(p,q)/L_t$. □

Remark 4.5.3. By its definition and this lemma, the function E is always nondecreasing. Actually, to be more precise, by means of a simple geometric argument it can be proved that if $E(t) = 1$ the curve γ_t must be a circle. Hence, in any other case E is strictly increasing in time.

An immediate consequence is that for every initial embedded, closed curve in $\mathbb{R}^2$, there exists a positive constant C depending on the initial curve such that on all $[0,T)$ we have $E(t) \geq C$. The same conclusion holds for any rescaling of such curves as the function E is scaling invariant by construction.

Remark 4.5.4. This lemma also provide an alternative proof of the fact that an initial embedded, closed curve stays embedded. Indeed, it cannot develop a self–intersection during its curvature flow, otherwise E would get zero.

We can then exclude type II singularities in the curvature flow of embedded closed curves. Any blow up limit flow γ^∞ is given (up to rigid motions) by a grim reaper, that is, the translating graph Γ of the function $y = -\log\cos x$ in the interval $(-\pi/2, \pi/2)$. Assuming that $\gamma_0^\infty = \Gamma$, we consider the following four points $p_1 = (-x_1, -\log\cos x_1)$, $q_1 = (x_1, -\log\cos x_1)$, $p_2 = (-x_2, -\log\cos x_2)$ and $q_2 = (x_2, -\log\cos x_2)$ belonging to Γ, for $0 < x_1 < x_2 < \pi/2$ such that $-\log\cos x_2 > \pi/2 - 3\log\cos x_1$.

As the rescaled curves γ_0^k converge locally in C^1 to Γ, for any $\varepsilon > 0$ such that $x_2 + \varepsilon < \pi/2$ and k is large enough the curve γ_0^k will be C^1–close to Γ in the open rectangle $R_\varepsilon = (-x_2 - \varepsilon, x_2 + \varepsilon) \times (-\varepsilon, -\log\cos x_2 + \varepsilon)$, hence there will be a pair of points $(p,q) \in \gamma_0^k$ arbitrarily close to (p_1, q_1) and another pair $(\widetilde{p}, \widetilde{q}) \in \gamma_0^k$ arbitrarily close to (p_2, q_2). As $k \to \infty$, the geodesic distance $d_{\gamma_0^k}(p,q)$ on the closed curve γ_0^k between p and q is definitely given by the length of the part of the curve which is close to the vertex of Γ, indeed, this latter is smaller than $\pi - 2\log\cos x_1$, when k is large enough, instead the *other part* of the curve has a length which is at least the sum of the Euclidean distances $|\widetilde{p} - p| + |\widetilde{q} - q|$ which is definitely larger than $2(\log\cos x_1 - \log\cos x_2)$ and this last quantity is larger than $\pi - 4\log\cos x_1$, by construction.

Hence, when k is large enough, the Huisken's embeddedness ratio for the rescaled curve γ_0^k is not larger than

$$\begin{aligned}\frac{\pi|p-q|}{L}\Big/\sin\frac{\pi d_{\gamma_0^k}(p,q)}{L} &\leq \frac{\pi(\pi+2\varepsilon)}{L}\Big/\sin\frac{\pi d_{\gamma_0^k}(p,q)}{L}\\ &\leq \frac{\pi(\pi+2\varepsilon)}{L}\Big/\frac{2d_{\gamma_0^k}(p,q)}{L}\\ &= \frac{\pi(\pi+2\varepsilon)}{2d_{\gamma_0^k}(p,q)}\\ &\leq \frac{\pi^2}{d_{\gamma_0^k}(p,q)},\end{aligned}$$

where L is the total length of the curve γ_0^k and we used the inequality $\sin x \geq 2x/\pi$ holding for every $x \in [0, \pi/2]$.

Finally, again by the C^1–convergence of γ_0^k to Γ in R_ε, we can also assume that $d_{\gamma_0^k}(p,q)$ is larger than $-\log\cos x_1$.

Now we consider a sequence of pairs $x_1^i < x_2^i$ as above such that $x_1^i \to \pi/2$, then we have a sequence of rescaled curves $\gamma_0^{k_i}$ such that the associated Huisken's embeddedness ratio tends to zero, as $d_{\gamma_0^{k_i}}(p,q) \to +\infty$ when $i \to \infty$.
This is in contradiction with the fact that the function E is scaling invariant and uniformly bounded from below by some positive constant C for all the curves of the flow.
As this argument does not change if we apply to Γ any rigid motion, in presence of a type II singularity in the embedded case, we would have a contradiction with the conclusion of Theorem 4.4.6.

Theorem 4.5.5. *Type II singularities cannot develop during the curvature flow of an embedded, closed curve in* $\mathbb{R}^2$.

Collecting together Theorem 3.4.1 about type I singularities and this last proposition, we obtain Theorem 3.3.7 by Gage and Hamilton and the following theorem due to Grayson [51], whose original proof is more geometric and direct, showing that the intervals of negative curvature vanish in finite time before any singularity. We underline that the success of the line of proof we followed is due to the bound from below on Huisken's embeddedness ratio implied by Lemma 4.5.2. Modifying a little such quantity, Andrews and Bryan [12] were even able to give a simple and direct proof without passing through the classification of singularities.

Theorem 4.5.6 (Grayson's Theorem). *Let γ_t be the curvature flow of a closed, embedded, smooth curve in the plane, in the maximal interval of smooth existence $[0,T)$.*
Then, there exists a time $\tau < T$ such that γ_τ is convex.
As a consequence, the result of Gage and Hamilton 3.3.7 applies and subsequently the curve shrinks smoothly to a point as $t \to T$.

Remark 4.5.7. This result, extended by Grayson to curves moving inside general surfaces, allowed him to have a proof of the *three geodesics theorem* on the sphere [53] (first outlined by Lusternik and Schnirelman in [89]).

We add a final remark in this case of embedded closed curves.
Letting $A(t)$ be the area enclosed by γ_t which moves by curvature, we have

$$\frac{d}{dt}A(t) = -\int_{\gamma_t} k\,ds = -2\pi\,,$$

hence, as the evolution is smooth till the curve shrinks to a point at time $T > 0$ and clearly $A(t)$ goes to zero, we have $A(0) = 2\pi T$. That is, the maximal time of existence is exactly equal to the initially enclosed area divided by 2π.