Lecture 3

Monotonicity Formula and Type I Singularities

In all this lecture $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ is the mean curvature flow of an n–dimensional, compact hypersurface in the maximal interval of smooth existence $[0,T)$.

As before we will use the notation $\varphi_t = \varphi(\,\cdot\,,t)$ and $\widetilde{\mathcal{H}}^n$ will be the n–dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ *counting multiplicities*.

3.1 The Monotonicity Formula for Mean Curvature Flow

We show the fundamental monotonicity formula for mean curvature flow, discovered by Huisken in [67] and then generalized by Hamilton in [58, 59].

Lemma 3.1.1. *Let $f : \mathbb{R}^{n+1} \times I \to \mathbb{R}$ be a smooth function. By a little abuse of notation, we denote by $\int_M f\,d\mu_t$ the integral $\int_M f(\varphi(p,t),t)\,d\mu_t(p)$.*
Then the following formula holds

$$\frac{d}{dt}\int_M f\,d\mu_t = \int_M (f_t - \mathrm{H}^2 f + \mathrm{H}\langle \nabla f \,|\, \nu\rangle)\,d\mu_t\,.$$

Proof. Straightforward computation. ☐

If $u : \mathbb{R}^{n+1} \times [0,\tau) \to \mathbb{R}$ is a smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$, by this lemma, we have

$$\begin{aligned}\frac{d}{dt}\int_M u\,d\mu_t &= \int_M (u_t - \mathrm{H}^2 u + \mathrm{H}\langle \nabla u \,|\, \nu\rangle)\,d\mu_t \\ &= -\int_M (\Delta^{\mathbb{R}^{n+1}} u + \mathrm{H}^2 u - \mathrm{H}\langle \nabla u \,|\, \nu\rangle)\,d\mu_t\,.\end{aligned} \tag{3.1.1}$$

Lemma 3.1.2. *If $\psi : M \to \mathbb{R}^{n+1}$ is a smooth isometric immersion of an n–dimensional Riemannian manifold (M,g), for every smooth function u defined in a neighborhood of $\psi(M)$ we have,*

$$\Delta_g(u(\psi)) = (\Delta^{\mathbb{R}^{n+1}} u)(\psi) - (\nabla^2_{\nu\nu} u)(\psi) + \mathrm{H}\langle (\nabla u)(\psi) \,|\, \nu\rangle\,,$$

where $(\nabla^2_{\nu\nu} u)(\psi(p))$ is the second derivative of u in the normal direction $\nu(p) \in \mathbb{R}^{n+1}$ at the point $\psi(p)$.

Proof. Let $p \in M$ and choose normal coordinates at p. Then,

$$\begin{aligned}
\Delta_g(u(\psi)) &= \nabla^2_{ii}(u(\psi)) \\
&= \nabla_i \left(\frac{\partial u}{\partial y_\alpha}(\psi)\, \frac{\partial \psi^\alpha}{\partial x_i} \right) \\
&= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}(\psi)\, \frac{\partial \psi^\alpha}{\partial x_i}\, \frac{\partial \psi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha}(\psi)\, \frac{\partial^2 \psi^\alpha}{\partial x_i^2} \\
&= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}(\psi)\, \frac{\partial \psi^\alpha}{\partial x_i}\, \frac{\partial \psi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha}(\psi)\, h_{ii} \nu^\alpha \\
&= (\Delta^{\mathbb{R}^{n+1}} u)(\psi) - (\nabla^2_{\nu\nu} u)(\psi) + \mathrm{H}\langle (\nabla u)(\psi) \,|\, \nu \rangle\,,
\end{aligned}$$

where we used the Gauss–Weingarten relations (1.1.1). □

It follows that, substituting $\Delta^{\mathbb{R}^{n+1}} u$ in formula (3.1.1) and using the previous lemma, if the function u is positive we get

$$\begin{aligned}
\frac{d}{dt} \int_M u \, d\mu_t &= - \int_M (\Delta_{g(t)}(u(\varphi_t)) + \nabla^2_{\nu\nu} u + \mathrm{H}^2 u - 2\mathrm{H}\langle \nabla u \,|\, \nu \rangle) \, d\mu_t \\
&= - \int_M (\nabla^2_{\nu\nu} u + \mathrm{H}^2 u - 2\mathrm{H}\langle \nabla u \,|\, \nu \rangle) \, d\mu_t \\
&= - \int_M \left| \mathrm{H} - \frac{\langle \nabla u \,|\, \nu \rangle}{u} \right|^2 u \, d\mu_t + \int_M \left(\frac{|\nabla^\perp u|^2}{u} - \nabla^2_{\nu\nu} u \right) d\mu_t\,,
\end{aligned}$$

where $\nabla^\perp u$ denotes the projection on the normal space to M of the gradient of u.
Then, assuming that $u : \mathbb{R}^{n+1} \times [0, \tau) \to \mathbb{R}$ is a positive smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ for some $\tau > 0$, the following formula easily follows,

$$\begin{aligned}
\frac{d}{dt} \Big[\sqrt{4\pi(\tau - t)} \int_M u \, d\mu_t \Big] = &- \sqrt{4\pi(\tau - t)} \int_M |\mathrm{H} - \langle \nabla \log u \,|\, \nu \rangle|^2 u \, d\mu_t \\
&- \sqrt{4\pi(\tau - t)} \int_M \Big(\nabla^2_{\nu\nu} u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau - t)} \Big) d\mu_t
\end{aligned} \tag{3.1.2}$$

in the time interval $[0, \min\{\tau, T\})$.
As we can see, the right hand side consists of a nonpositive quantity and a term which is nonpositive if $\frac{\nabla^2_{\nu\nu} u}{u} - \frac{|\nabla^\perp u|^2}{u^2} + \frac{1}{2(\tau - t)} = \nabla^2_{\nu\nu} \log u + \frac{1}{2(\tau - t)}$ is nonnegative.
Setting $v(x, s) = u(x, \tau - s)$, the function $v : \mathbb{R}^{n+1} \times (0, \tau] \to \mathbb{R}$ is a positive solution of the standard *forward* heat equation in all $\mathbb{R}^{n+1}$ and setting $t = \tau - s$ we have $\nabla^2_{\nu\nu} \log u + \frac{1}{2(\tau - t)} = \nabla^2_{\nu\nu} \log v + \frac{1}{2s}$.
This last expression is exactly the Li–Yau–Hamilton 2–form $\nabla^2 \log v + g/(2s)$ for positive solutions of the heat equation on a compact manifold (M, g), evaluated on $\nu \otimes \nu$ (see [58]).
In the paper [58] (see also [93]) Hamilton generalized the Li–Yau differential Harnack inequality in [86] (concerning the nonnegativity of $\Delta \log v + \frac{\dim M}{2s}$) showing that, under the assumptions that the compact manifold (M, g) has parallel Ricci tensor ($\nabla \mathrm{Ric} = 0$) and nonnegative sectional curvatures, the 2–form $\nabla^2 \log v + g/(2s)$ is nonnegative definite (Hamilton's matrix Li–Yau–Harnack inequality). Even if it is not compact, this result also holds in $\mathbb{R}^{n+1}$ with the canonical flat metric (which clearly satisfies the above hypotheses on the curvature), assuming the boundedness in space of the function v (equivalently of u), at every fixed time. Hence, $\nabla^2_{\nu\nu} \log u + \frac{1}{2(\tau - t)} = \Big(\nabla^2 \log v + g^{\mathbb{R}^{n+1}}_{\mathrm{can}} / (2s) \Big)(\nu \otimes \nu) \geq 0$. It follows that, if a smooth solution u of the backward heat equation is bounded in space at every fixed time, the monotonicity formula implies that $\sqrt{4\pi(\tau - t)} \int_M u \, d\mu_t$ is nonincreasing in time.
We resume this discussion in the following theorem by Hamilton [58, 59].

Theorem 3.1.3 (Huisken's Monotonicity Formula – Hamilton's Extension in $\mathbb{R}^{n+1}$)**.** *Assume that for some $\tau > 0$ we have a positive smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ in $\mathbb{R}^{n+1} \times [0, \tau)$, bounded in space for every fixed $t \in [0, \tau)$, then*

$$\frac{d}{dt}\left[\sqrt{4\pi(\tau - t)}\int_M u\,d\mu_t\right] \leq -\sqrt{4\pi(\tau - t)}\int_M |\mathrm{H} - \langle \nabla \log u \,|\, \nu\rangle|^2 u\,d\mu_t$$

in the time interval $[0, \min\{\tau, T\})$.

Choosing in particular a *backward* heat kernel of $\mathbb{R}^{n+1}$, that is,

$$u(x, t) = \rho_{x_0,\tau}(x, t) = \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{(n+1)/2}}$$

in formula (3.1.2), we get the standard Huisken's monotonicity formula, as the Li–Yau–Hamilton expression $\nabla^2_{\nu\nu} u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau-t)}$ is identically zero in this case.

Theorem 3.1.4 (Huisken's Monotonicity Formula)**.** *For every $x_0 \in \mathbb{R}^{n+1}$ and $\tau > 0$ we have (see [67])*

$$\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\,d\mu_t = -\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x - x_0 \,|\, \nu\rangle}{2(\tau - t)}\right|^2 d\mu_t$$

in the time interval $[0, \min\{\tau, T\})$.
Hence, the integral $\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t$ is nonincreasing during the flow in $[0, \min\{\tau, T\})$.

Exercise 3.1.5. Show that for every $x_0 \in \mathbb{R}^{n+1}$, $\tau > 0$ and a smooth function $v : M \times [0, T) \to \mathbb{R}$, we have

$$\begin{aligned}\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\,v\,d\mu_t = &-\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x - x_0 \,|\, \nu\rangle}{2(\tau - t)}\right|^2 v\,d\mu_t \\ &+\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\,(v_t - \Delta_{g(t)} v)\,d\mu_t\,,\end{aligned}$$

in the time interval $[0, \min\{\tau, T\})$.
In particular if $v : M \times [0, T) \to \mathbb{R}$ is a smooth solution of $v_t = \Delta_{g(t)} v$, it follows

$$\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\,v\,d\mu_t = -\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x - x_0 \,|\, \nu\rangle}{2(\tau - t)}\right|^2 v\,d\mu_t$$

in $[0, \min\{\tau, T\})$.

3.2 Type I Singularities and the Rescaling Procedure

In the previous lecture we showed that the curvature must blow up at the maximal time T with the following lower bound

$$\max_{p\in M} |\mathrm{A}(p, t)| \geq \frac{1}{\sqrt{2(T - t)}}\,.$$

Definition 3.2.1. Let T be the maximal time of existence of a mean curvature flow. If there exists a constant $C > 1$ such that we have the upper bound

$$\max_{p\in M} |\mathrm{A}(p, t)| \leq \frac{C}{\sqrt{2(T - t)}}\,,$$

we say that the flow is developing at time T a *type I singularity*.
If such a constant does not exist, that is,

$$\limsup_{t\to T}\max_{p\in M}|\mathrm{A}(p,t)|\sqrt{T-t}=+\infty$$

we say that we have a *type II singularity*.

In this lecture we will deal exclusively with type I singularities and the monotonicity formula will be the main tool for the analysis. The next lecture will be devoted to type II singularities.

From now on, we assume that there exists some constant $C_0 > 1$ such that

$$\frac{1}{\sqrt{2(T-t)}}\leq\max_{p\in M}|\mathrm{A}(p,t)|\leq\frac{C_0}{\sqrt{2(T-t)}}\,, \tag{3.2.1}$$

for every $t\in[0,T)$.

Let $p\in M$ and $0\leq t\leq s<T$, then

$$|\varphi(p,s)-\varphi(p,t)|=\left|\int_t^s\frac{\partial\varphi(p,\xi)}{\partial t}\,d\xi\right|\leq\int_t^s|\mathrm{H}(p,\xi)|\,d\xi\leq\int_t^s\frac{C_0\sqrt{n}}{\sqrt{2(T-\xi)}}\,d\xi\leq C_0\sqrt{n(T-t)}$$

which implies that the sequence of functions $\varphi(\,\cdot\,,t)$ converges as $t\to T$ to some function $\varphi_T : M\to\mathbb{R}^{n+1}$. Moreover, as the constant C_0 is independent of $p\in M$, such convergence is uniform and the limit function φ_T is continuous. Finally, passing to the limit in the above inequality, we get

$$|\varphi(p,t)-\varphi_T(p)|\leq C_0\sqrt{n(T-t)}\,. \tag{3.2.2}$$

In all the lecture we will denote $\varphi_T(p)$ also by $\widehat{p}$.

Definition 3.2.2. Let $\mathcal{S}$ be the set of points $x\in\mathbb{R}^{n+1}$ such that there exists a sequence of pairs $(p_i,t_i)\in M\times[0,T)$ with $t_i\nearrow T$ and $\varphi(p_i,t_i)\to x$.
We call $\mathcal{S}$ the set of *reachable* points.

We have seen in Proposition 2.2.6 that $\mathcal{S}$ is compact and that $x\in\mathcal{S}$ if and only if, for every $t\in[0,T)$ the closed ball of radius $\sqrt{2n(T-t)}$ and center x intersects $\varphi(M,t)$. We show now that $\mathcal{S}=\{\widehat{p}\,|\,p\in M\}$.
Clearly $\{\widehat{p}\,|\,p\in M\}\subset\mathcal{S}$, suppose that $x\in\mathcal{S}$ and $\varphi(p_i,t_i)\to x$, then, by inequality (3.2.2) we have $|\varphi(p_i,t_i)-\widehat{p}_i|\leq C_0\sqrt{n(T-t_i)}$, hence, $\widehat{p}_i\to x$ as $i\to\infty$. As the set $\{\widehat{p}\,|\,p\in M\}$ is closed it follows that it must contain the point x.

We define now a tool which will be fundamental in the sequel.

Definition 3.2.3. For every $p\in M$, we define the *heat density* function

$$\theta(p,t)=\int_M\frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t$$

and the *limit heat density* function

$$\Theta(p)=\lim_{t\to T}\theta(p,t)\,.$$

Since M is compact, we can also define the following *maximal heat density* function

$$\sigma(t)=\max_{x_0\in\mathbb{R}^{n+1}}\int_M\frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t \tag{3.2.3}$$

and its limit $\Sigma=\lim_{t\to T}\sigma(t)$.

Clearly, $\theta(p,t) \leq \sigma(t)$ for every $p \in M$ and $t \in [0,T)$ and $\Theta(p) \leq \Sigma$ for every $p \in M$. The function Θ is well defined as the limit exists finite since $\theta(p,t)$ is monotone nonincreasing in t and positive. Moreover, the functions $\theta(\,\cdot\,,t)$ are all continuous and monotonically converging to Θ, hence this latter is upper semicontinuous and nonnegative.

The function $\sigma : [0,T) \to \mathbb{R}$ is also positive and monotone nonincreasing, being the maximum of a family of nonincreasing smooth functions, hence the limit Σ is well defined and finite. Moreover, such family is uniformly locally Lipschitz (look at the right hand side of the monotonicity formula), hence also σ is locally Lipschitz, then by Hamilton's trick 2.1.3, at every differentiability time $t \in [0,T)$ of σ we have the following *maximal* monotonicity formula

$$\sigma'(t) = -\int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \left| \mathrm{H} + \frac{\langle x - x_t \,|\, \nu\rangle}{2(T-t)} \right|^2 d\mu_t \tag{3.2.4}$$

where $x_t \in \mathbb{R}^{n+1}$ is any point where the maximum defining $\sigma(t)$ is attained, that is,

$$\sigma(t) = \int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t\,.$$

Remark 3.2.4. Notice that we did not define $\sigma(t)$ as the maximum of $\theta(\,\cdot\,,t)$

$$\max_{p\in M} \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t$$

which is *taken among* $p \in M$. Clearly, this latter can be smaller than $\sigma(t)$.

For any point $p \in M$, we rescale now the moving hypersurfaces around $\widehat{p} = \lim_{t\to T} \varphi(p,t)$, following Huisken [67],

$$\widetilde{\varphi}(q,s) = \frac{\varphi(q,t(s)) - \widehat{p}}{\sqrt{2(T-t(s))}} \qquad s = s(t) = -\frac{1}{2}\log(T-t)$$

and we compute the evolution equation for the map $\widetilde{\varphi}(q,s)$ in the time interval $\left[-\frac{1}{2}\log T, +\infty\right)$,

$$\begin{aligned}
\frac{\partial\widetilde{\varphi}(q,s)}{\partial s} &= \left(\frac{ds}{dt}\right)^{-1} \frac{\partial}{\partial t}\left(\frac{\varphi(q,t) - \widehat{p}}{\sqrt{2(T-t)}}\right)\\
&= \sqrt{2(T-t)}\,\frac{\partial\varphi(q,t)}{\partial t} + \frac{\varphi(q,t)-\widehat{p}}{\sqrt{2(T-t)}}\\
&= \sqrt{2(T-t)}\,\mathrm{H}(q,t)\nu(q,t) + \widetilde{\varphi}(q,s)\\
&= \widetilde{\mathrm{H}}(q,s)\widetilde{\nu}(q,s) + \widetilde{\varphi}(q,s)\,,
\end{aligned}$$

where $\widetilde{\mathrm{H}}$ is the mean curvature of the rescaled hypersurfaces $\widetilde{\varphi}_s = \widetilde{\varphi}(\,\cdot\,,s)$.
As $|\widetilde{\mathrm{A}}| = \sqrt{2(T-t)}\,|\mathrm{A}| \leq C_0 < +\infty$, all the hypersurfaces $\widetilde{\varphi}_s$ have equibounded curvatures, moreover,

$$|\widetilde{\varphi}(p,s)| = \left|\frac{\varphi(p,t(s)) - \widehat{p}}{\sqrt{2(T-t(s))}}\right| \leq \frac{C_0\sqrt{2n(T-t(s))}}{\sqrt{2(T-t(s))}} = C_0\sqrt{n} \tag{3.2.5}$$

which implies that at every time $s \in \left[-\frac{1}{2}\log T, +\infty\right)$ the open ball of radius $C_0\sqrt{2n}$ centered at the origin of $\mathbb{R}^{n+1}$ intersects the hypersurface $\widetilde{\varphi}(\,\cdot\,,s)$. More precisely, the point $\widetilde{\varphi}(p,s)$ belongs to the interior of such ball.

Then, we rescale also the monotonicity formula. In the following $\widetilde{\mu}_s = \frac{\mu_t}{[2(T-t)]^{n/2}}$ will be the canonical measure associated to the rescaled hypersurface $\widetilde{\varphi}_s$ which, by means of equation (2.3.1), satisfies

$$\frac{d}{ds}\widetilde{\mu}_s = (n - \widetilde{\mathrm{H}}^2)\widetilde{\mu}_s\,,$$

as

$$\begin{aligned}
\frac{\partial}{\partial s}\widetilde{\mu}_s &= \left(\frac{ds}{dt}\right)^{-1}\frac{\partial}{\partial t}\left(\frac{\mu_t}{[2(T-t)]^{n/2}}\right)\\
&= n\left(\frac{\mu_t}{[2(T-t)]^{n/2}}\right) + \frac{1}{[2(T-t)]^{n/2-1}}\frac{\partial}{\partial t}\mu_t\\
&= n\widetilde{\mu}_s - \frac{1}{[2(T-t)]^{n/2-1}}\mathrm{H}^2\mu_t\\
&= n\widetilde{\mu}_s - \widetilde{\mathrm{H}}^2\widetilde{\mu}_s\,.
\end{aligned}$$

Proposition 3.2.5 (Rescaled Monotonicity Formula)**.** *We have*

$$\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s = -\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_s \leq 0\,, \tag{3.2.6}$$

which integrated becomes

$$\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{s_1} - \int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{s_2} = \int_{s_1}^{s_2}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_s\,ds\,.$$

In particular,

$$\int_{-\frac{1}{2}\log T}^{+\infty}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_s\,ds \leq \int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{-\frac{1}{2}\log T} \leq C < +\infty\,,$$

for a uniform constant $C = C(\mathrm{Area}(\varphi_0), T)$ independent of $s \in \left[-\frac{1}{2}\log T, +\infty\right)$ and $p \in M$.

Proof. Keeping in mind that $y = \frac{x-\widehat{p}}{\sqrt{2(T-t)}}$ and $s = -\frac{1}{2}\log(T-t)$ we have,

$$\begin{aligned}
\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s &= \left(\frac{ds}{dt}\right)^{-1}\frac{d}{dt}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s\\
&= 2(T-t)\frac{d}{dt}\int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[2(T-t)]^{n/2}}\,d\mu_t\\
&= -\,2(T-t)\int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[2(T-t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x-\widehat{p}\,|\,\nu\rangle}{2(T-t)}\right|^2\,d\mu_t\\
&= -\,2(T-t)\int_M e^{-\frac{|y|^2}{2}}\left|\frac{\widetilde{\mathrm{H}}}{\sqrt{2(T-t)}} + \frac{\langle y\,|\,\widetilde{\nu}\rangle}{\sqrt{2(T-t)}}\right|^2\,d\widetilde{\mu}_s\\
&= -\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_s\,.
\end{aligned}$$

The other two statements trivially follow. □

As a first consequence, we work out an upper estimate on the volume of the rescaled hypersurfaces in the balls of $\mathbb{R}^{n+1}$.

Fix a radius $R > 0$, if $B_R = B_R(0) \subset \mathbb{R}^{n+1}$, then we have

$$\begin{aligned}\widetilde{\mathcal{H}}^n(\widetilde{\varphi}(M,s) \cap B_R) &= \int_M \chi_{B_R}(y)\, d\widetilde{\mu}_s \\ &\leq \int_M \chi_{B_R}(y) e^{\frac{R^2-|y|^2}{2}}\, d\widetilde{\mu}_s \\ &\leq e^{R^2/2} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_s \\ &\leq e^{R^2/2} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{-\frac{1}{2}\log T} \\ &\leq \widehat{C} e^{R^2/2}\end{aligned} \tag{3.2.7}$$

where the constant $\widehat{C}$ is independent of R and s.

Remark 3.2.6. As

$$\int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{-\frac{1}{2}\log T} = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4T}}}{(2T)^{n/2}}\, d\mu_0 \leq \frac{\mathrm{Area}(\varphi_0)}{(2T)^{n/2}}\,,$$

we can choose the constant $\widehat{C}$ to be independent also of $p \in M$.

Another consequence is the following key technical lemma which is necessary in order to take the limits of integrals of functions on the sequences of rescaled hypersurfaces.

Lemma 3.2.7 (Stone [115])**.** *The following estimates hold.*

1. *There is a uniform constant $C = C(n, \mathrm{Area}(\varphi_0), T)$ such that, for any $p \in M$ and for all $s \in \left[-\frac{1}{2}\log T, +\infty\right)$,*

$$\int_M e^{-|y|}\, d\widetilde{\mu}_s \leq C\,.$$

2. *For any $\varepsilon > 0$ there is a uniform radius $R = R(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$ such that, for any $p \in M$ and for all $s \in \left[-\frac{1}{2}\log T, +\infty\right)$,*

$$\int_{\widetilde{\varphi}_s(M)\setminus B_R(0)} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n \leq \varepsilon\,,$$

that is, the family of measures $e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n \llcorner \widetilde{\varphi}_s(M)$ is tight *(see [32]).*

Proof. By the rescaled monotonicity formula (3.2.6) we have that, for any $p \in M$ and for all $s \in \left[-\frac{1}{2}\log T, +\infty\right)$,

$$\int_M e^{-|y|^2/2}\, d\widetilde{\mu}_s \leq \int_M e^{-|y|^2/2}\, d\widetilde{\mu}_{-\frac{1}{2}\log T}\,.$$

According to Remark 3.2.6, the right hand integral may be estimated by a constant depending only on T and $\mathrm{Area}(\varphi_0)$, not on $p \in M$. Hence, we have the following estimates for all $p \in M$ and for all $s \in \left[-\frac{1}{2}\log T, +\infty\right)$,

$$\int_{\widetilde{\varphi}_s(M)\cap B_{n+1}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_1 \tag{3.2.8}$$

and

$$\int_{\widetilde{\varphi}_s(M)\cap B_{2n+2}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_2 \tag{3.2.9}$$

where C_1 and C_2 are constants depending only on n, T and $\mathrm{Area}(\varphi_0)$.
Then, we compute for any p and s,

$$\begin{aligned}\frac{d}{ds}\int_M e^{-|y|}\,d\widetilde{\mu}_s &= \int_M \Big\{n-\widetilde{\mathrm{H}}^2-\frac{1}{|y|}\langle y\,|\,\widetilde{\mathrm{H}}\widetilde{\nu}+y\rangle\Big\}e^{-|y|}\,d\widetilde{\mu}_s\\ &\leq \int_M \big\{n-\widetilde{\mathrm{H}}^2-|y|+|\widetilde{\mathrm{H}}|\big\}e^{-|y|}\,d\widetilde{\mu}_s\\ &< \int_M \big\{n+1-|y|\big\}e^{-|y|}\,d\widetilde{\mu}_s\\ &\leq (n+1)\left\{\int_{\widetilde{\varphi}_s(M)\cap B_{n+1}(0)} e^{-|y|}\,d\widetilde{\mathcal{H}}^n-\int_{\widetilde{\varphi}_s(M)\setminus B_{2n+2}(0)} e^{-|y|}\,d\widetilde{\mathcal{H}}^n\right\}.\end{aligned}$$

But then, by inequality (3.2.8) we see that we must have either

$$\frac{d}{ds}\int_M e^{-|x|}\,d\widetilde{\mu}_s<0\,,$$

or

$$\int_{\widetilde{\varphi}_s(M)\setminus B_{2n+2}(0)} e^{-|y|}\,d\widetilde{\mathcal{H}}^n\leq C_1\,.$$

Hence, in view of inequality (3.2.9), it follows that either

$$\frac{d}{ds}\int_M e^{-|y|}\,d\widetilde{\mu}_s<0\,,$$

or

$$\int_M e^{-|y|}\,d\widetilde{\mu}_s\leq C_1+C_2\,,$$

which implies

$$\int_M e^{-|y|}\,d\widetilde{\mu}_s\leq \max\left\{(C_1+C_2),\int_M e^{-|y|}\,d\widetilde{\mu}_{-\frac{1}{2}\log T}\right\}=C_3$$

for any p and s.
The proof of part (1) of the lemma follows by noticing that the integral quantity on the right hand side can clearly be estimated by a constant depending on T and $\mathrm{Area}(\varphi_0)$ but not on $p\in M$.

Let now again $p\in M$ and $s\in\left[-\frac{1}{2}\log T,+\infty\right)$ arbitrary. Now subdivide $\widetilde{\varphi}_s(M)$ into "annular pieces" $\{\widetilde{M}_s^k\}_{k=0}^{\infty}$ by setting

$$\widetilde{M}_s^0=\widetilde{\varphi}_s(M)\cap B_1(0)\,,$$

and for each $k\geq 1$,

$$\widetilde{M}_s^k=\big\{y\in\widetilde{\varphi}_s(M)\,|\,2^{k-1}\leq|y|<2^k\big\}\,.$$

Then, by part (1) of the lemma $\widetilde{\mathcal{H}}^n(\widetilde{M}_s^k)\leq C_3e^{(2^k)}$ for each k, independently of the choice of p and s. Hence in turn, for each k we have

$$\int_{\widetilde{M}_s^k} e^{-|y|^2/2}\,d\widetilde{\mathcal{H}}^n\leq C_3e^{-\frac{1}{2}(2^{k-1})^2}e^{(2^k)}=C_3e^{(2^k-2^{2k-3})}$$

again independently of the choice of p and s.
For any $\varepsilon>0$ we can find a $k_0=k_0(\varepsilon,n,\mathrm{Area}(\varphi_0),T)$ such that

$$\sum_{k=k_0}^{\infty}C_3e^{(2^k-2^{2k-3})}\leq\varepsilon\,,$$

then, if $R = R(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$ is simply taken to be equal to 2^{k_0-1}, we have

$$\int_{\widetilde{\varphi}_s(M)\setminus B_R(0)} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n = \sum_{k=k_0}^{\infty} \int_{\widetilde{M}_s^k} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n \le \sum_{k=k_0}^{\infty} C_3 e^{(2^k - 2^{2k-3})} \le \varepsilon$$

and we are done also with part (2) of the lemma. □

Corollary 3.2.8. *If a sequence of rescaled hypersurfaces $\widetilde{\varphi}_{s_i}$ locally smoothly converges (up to reparametrization) to some limit hypersurface $\widetilde{M}_\infty$, we have*

$$\int_{\widetilde{M}_\infty} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \le C$$

and

$$\lim_{i\to\infty} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{s_i} = \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n\,,$$

where the constant C is the same of the previous lemma.

Proof. Actually, it is only sufficient to show that the measures $\widetilde{\mathcal{H}}^n \llcorner \widetilde{\varphi}(M, s_i)$ associated to the hypersurfaces weakly*–converge to the measure $\widetilde{\mathcal{H}}^n \llcorner \widetilde{M}_\infty$. Indeed, for every $R > 0$ we have,

$$\int_{\widetilde{M}_\infty \cap B_R(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \le \liminf_{i\to\infty} \int_{\widetilde{\varphi}(M,s_i)\cap B_R(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \le \liminf_{i\to\infty} \int_M e^{-|y|}\, d\widetilde{\mu}_{s_i} \le C$$

by the first part of the lemma above. Sending R to $+\infty$, the first inequality follows.
The second statement is an easy consequence of the estimates in the second part of the lemma. □

Now we want to estimate the covariant derivatives of the rescaled hypersurfaces.

Proposition 3.2.9 (Huisken [67]). *For every $k \in \mathbb{N}$ there exists a constant C_k depending only on k, n, C_0 (the constant in formula* (3.2.1)*) and the initial hypersurface such that $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}} \le C_k$ for every $p \in M$ and $s \in \left[-\frac{1}{2}\log T, +\infty\right)$.*

Proof. By Proposition 2.3.5 we have for the original flow,

$$\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 = \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}\,,$$

hence, with a straightforward computation, noticing that $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} = |\nabla^k \mathrm{A}|^2_g [2(T-t)]^{k+1}$ we get

$$\begin{aligned}\frac{\partial}{\partial s}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \le &- 2(k+1)|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + \widetilde{\Delta}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\ &+ C(n,k) \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} |\widetilde{\nabla}^p \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^q \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^r \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}}\,.\end{aligned}$$

As $|\widetilde{\mathrm{A}}|_{\widetilde{g}}$ is bounded by the constant C_0, supposing by induction that for $i = 0, \dots, k-1$ we have uniform bounds on $|\widetilde{\nabla}^i \widetilde{\mathrm{A}}|_{\widetilde{g}}$ with constants $C_i = C_i(n, C_0, \varphi_0)$, we can conclude by means of Peter–Paul inequality

$$\frac{\partial}{\partial s}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \le \widetilde{\Delta}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + D_k$$

for some constants B_k and D_k depending only on n, k, C_0 and the initial hypersurface.
Then,

$$\begin{aligned}\frac{\partial}{\partial s}(|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}) \leq{}& \widetilde{\Delta}|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\
&+ B_k\widetilde{\Delta}|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_kB_{k-1}|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\
&- 2B_k|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + D_k + B_kD_{k-1}\\
\leq{}& \widetilde{\Delta}(|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}) - B_k|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\
&+ B_kB_{k-1}|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + D_k + B_kD_{k-1}\\
\leq{}& \widetilde{\Delta}(|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}) - B_k|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\
&+ B_kB_{k-1}C^2_{k-1} + D_k + B_kD_{k-1}\\
\leq{}& \widetilde{\Delta}(|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}})\\
&- B_k(|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}})\\
&+ B_kB_{k-1}C^2_{k-1} + D_k + B_kD_{k-1} + B^2_kC^2_{k-1}\end{aligned}$$

where we used the inductive hypothesis $|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_{k-1}$.
By the maximum principle, the function $|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}}+B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}$ is then uniformly bounded in space and time by the square of some constant C_k depending on n, k, C_0 and the initial hypersurface, hence $|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_k$. By the inductive hypothesis, the thesis of the proposition follows. □

We are now ready to study the convergence of the rescaled hypersurfaces as $s \to +\infty$.

Proposition 3.2.10. *For every point $p \in M$ and every sequence of times $s_i \to +\infty$ there exists a subsequence (not relabeled) of times such that the hypersurfaces $\widetilde{\varphi}_{s_i}$, rescaled around $\widehat{p}$, locally smoothly converge (up to reparametrization) to some nonempty, smooth, complete limit hypersurface $\widetilde{M}_\infty$ such that $\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle = 0$ for every $y \in \widetilde{M}_\infty$.*
Any limit hypersurface satisfies $\widetilde{\mathcal{H}}^n(\widetilde{M}_\infty \cap B_R) \leq C_R$ for every ball of radius R in $\mathbb{R}^{n+1}$ and for every $k \in \mathbb{N}$ there are constants C_k such that $|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_k$.
Moreover, if the initial hypersurface was embedded, $\widetilde{M}_\infty$ is embedded.

Proof. We give a sketch of the proof, following Huisken [67].
By estimate (3.2.7) there is a uniform upper bound on $\mathcal{H}^n(\widetilde{\varphi}(M,s) \cap B_R)$ for each R, independent of s. Moreover, by the uniform control on the norm of the second fundamental form of the rescaled hypersurfaces in Proposition 3.2.9, there is a number $r_0 > 0$ such that, for each $s \in \left[-\frac{1}{2}\log T, +\infty\right)$ and each $q \in M$, if $U^s_{r_0,q}$ is the connected component of $\widetilde{\varphi}_s^{-1}(B_{r_0}(\widetilde{\varphi}_s(q)))$ in M containing q, then $\widetilde{\varphi}_s(U^s_{r_0,q})$ can be written as a graph of a smooth function f over a subset of the ball of radius r_0 in the tangent hyperplane to $\widetilde{\varphi}_s(M) \subset \mathbb{R}^{n+1}$ at the point $\widetilde{\varphi}_s(q)$.
The estimates of Proposition 3.2.9 then imply that all the derivatives of such function f up to the order $\alpha \in \mathbb{N}$ are bounded by constants C_α independent of s.
Following now the method in [83] we can see that, for each $R > 0$, a subsequence of the hypersurfaces $\widetilde{\varphi}(M,s) \cap B_R(0)$ must converge smoothly to a limit hypersurface in $B_R(0)$. Then, the existence of a smooth, complete limit hypersurface $\widetilde{M}_\infty$ follows from a diagonal argument, letting $R \to +\infty$. Recalling the fact that every rescaled hypersurface intersects the ball of radius $C_0\sqrt{2n}$ centered at the origin of $\mathbb{R}^{n+1}$, this limit cannot be empty. The estimates on the volume and derivatives of the curvature follow from the analogous properties for the converging sequence.
The fact that $\widetilde{M}_\infty$ satisfies $\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle = 0$ is a consequence of the rescaled monotonicity formula and of the uniform estimates on the curvature and its covariant derivatives for the rescaled hy-

persurfaces in Proposition 3.2.9. Indeed, by means of equation (2.3.4) we have

$$\begin{aligned}\frac{\partial \widetilde{\mathrm{H}}}{\partial s} &= \left(\frac{ds}{dt}\right)^{-1} \frac{\partial}{\partial t}\Big(\sqrt{2(T-t)}\,\mathrm{H}\Big)\\ &= \Big(2(T-t)\Big)^{3/2}(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2) - \sqrt{2(T-t)}\,\mathrm{H}\\ &= \widetilde{\Delta}\widetilde{\mathrm{H}} + \widetilde{\mathrm{H}}|\widetilde{\mathrm{A}}|^2 - \widetilde{\mathrm{H}}\end{aligned}$$

and since $\widetilde{\nu} = \nu$,

$$\frac{\partial \widetilde{\nu}}{\partial s} = \left(\frac{ds}{dt}\right)^{-1}\frac{\partial \nu}{\partial t} = -2(T-t)\nabla\mathrm{H} = -\widetilde{\nabla}\widetilde{\mathrm{H}}\,.$$

Hence,

$$\begin{aligned}\left|\frac{\partial}{\partial s}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\right| &= 2\left|\Big(\widetilde{\Delta}\widetilde{\mathrm{H}} + \widetilde{\mathrm{H}}|\widetilde{\mathrm{A}}|^2 - \widetilde{\mathrm{H}} + \langle\widetilde{\mathrm{H}}\widetilde{\nu} + y\,|\,\widetilde{\nu}\rangle - \langle y\,|\,\widetilde{\nabla}\widetilde{\mathrm{H}}\rangle\Big)\Big(\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\Big)\right|\\ &= 2\left|\,\widetilde{\Delta}\widetilde{\mathrm{H}} + \widetilde{\mathrm{H}}|\widetilde{\mathrm{A}}|^2 + \langle y\,|\,\widetilde{\nu}\rangle - \langle y\,|\,\widetilde{\nabla}\widetilde{\mathrm{H}}\rangle\right|\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|\\ &\leq C(|y| + C)(|y| + C)\\ &\leq C(|y|^2 + 1)\end{aligned}$$

for some constant C independent of s.
Then,

$$\begin{aligned}&\left|\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\right| \qquad (3.2.10)\\ &= \left|\int_M e^{-\frac{|y|^2}{2}}\left[\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\Big(n - \widetilde{\mathrm{H}}^2 - \langle y\,|\,\widetilde{\mathrm{H}}\widetilde{\nu} + y\rangle\Big) + \frac{\partial}{\partial s}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\right] d\widetilde{\mu}_s\right|\\ &\leq \int_M e^{-\frac{|y|^2}{2}}\left[C(|y|^2+1)(|y|^2+1) + C(|y|^2+1)\right] d\widetilde{\mu}_s\\ &\leq C\int_M e^{-\frac{|y|^2}{2}}(|y|^4+1)\,d\widetilde{\mu}_s\end{aligned}$$

and this last term is bounded uniformly in $s \in \Big[-\frac{1}{2}\log T, +\infty\Big)$ by a positive constant $C = C(\mathrm{Area}(\varphi_0), T)$ by the estimates in Stone's Lemma 3.2.7.
Supposing that there is a sequence of times $s_i \to +\infty$ such that

$$\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{s_i} \geq \delta$$

for some $\delta > 0$, then we have that in all the intervals $[s_i, s_i + \delta/(2C))$ such integral is larger than $\delta/2$. This is clearly in contradiction with the fact that

$$\int_{-\frac{1}{2}\log T}^{+\infty}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\,ds < +\infty\,,$$

stated in Proposition 3.2.5.
If $\widetilde{\varphi}_{s_i}$ is a locally smoothly converging subsequence of rescaled hypersurfaces (up to reparametrization), we have then that for every ball B_R

$$\int_{\widetilde{\varphi}(M,s_i)\cap B_R} e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mathcal{H}}^n \leq \int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{s_i} \to 0\,,$$

hence, the limit hypersurface $\widetilde{M}_\infty$ satisfies $\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle = 0$ at all its points.

Assume now that the initial hypersurface was embedded, then by Proposition 2.2.7 all the hypersurfaces $\widetilde{\varphi}_s$ are embedded and the only possibility for $\widetilde{M}_\infty$ not to be embedded is that two or more of its regions "touch" each other at some point $y \in \mathbb{R}^{n+1}$ with a common tangent space. Let $g(t)$ be the metrics induced on the moving hypersurfaces, we consider the following set $\Omega_\varepsilon \subset M \times M \times [0,T)$ given by $\{(p,q,t) \,|\, d_t(p,q) \leq \varepsilon\sqrt{2(T-t)}\}$, where d_t is the geodesic distance in the Riemannian manifold $(M, g(t))$. Let

$$B_\varepsilon = \inf_{\partial\Omega_\varepsilon} |\varphi(p,t) - \varphi(q,t)|/\sqrt{2(T-t)}\,,$$

we claim that $B_\varepsilon > 0$ for any $\varepsilon > 0$ small enough. Suppose that $B_\varepsilon = 0$ for some $\varepsilon > 0$, this means that there exists a sequence of times $t_i \nearrow T$ and points p_i, q_i with $d_{t_i}(p_i, q_i) = \varepsilon\sqrt{2(T-t_i)}$ and $|\varphi(p_i,t_i) - \varphi(q_i,t_i)|/\sqrt{2(T-t_i)} \to 0$, hence, $|\widetilde{\varphi}_i(p_i) - \widetilde{\varphi}_i(q_i)| \to 0$ and $\widetilde{d}_i(p_i,q_i) = \varepsilon$, where we denoted by $\widetilde{\varphi}_i$ the rescaled hypersurfaces $\widetilde{\varphi}_i(p) = \frac{\varphi(p,t_i)-\varphi(p_i,t_i)}{\sqrt{2(T-t_i)}}$ and $\widetilde{d}_i = d_{t_i}/\sqrt{2(T-t_i)}$.
Reasoning like in the first part of this proof, by the uniform bound on the second fundamental form of the rescaled hypersurfaces, if U^i is the connected component of $\widetilde{\varphi}_i^{-1}(B_{r_0}(\widetilde{\varphi}_i(p_i)))$ containing p_i, then $\widetilde{\varphi}_i(U^i)$ can be written as a graph of a smooth function f_i over a subset of the tangent hyperplane to $\widetilde{\varphi}_i(M) \subset \mathbb{R}^{n+1}$ at the point $\widetilde{\varphi}_i(p_i)$. As $\widetilde{d}_i(p_i,q_i) = \varepsilon$, when $\varepsilon > 0$ is small enough (depending on r_0 and C_0) the Lipschitz constants of these functions f_i are uniformly bounded by a constant depending on r_0 and C_0, moreover, for every $i \in \mathbb{N}$ the point q_i stays in U^i and $\widetilde{\varphi}_i(q_i)$ belongs to the graph of f_i.
It is then easy to see that there exists a uniform positive bound from below on $|\widetilde{\varphi}_i(p_i) - \widetilde{\varphi}_i(q_i)|$, hence the constant B_ε cannot be zero for such $\varepsilon > 0$.
Supposing that $\widetilde{M}_\infty$ has a self–intersection, we can parametrize it locally with a map $\widetilde{\varphi}_\infty : U \to \mathbb{R}^{n+1}$ such that a sequence of reparametrizations of the rescaled hypersurfaces $\widetilde{\varphi}_i$ converges smoothly to $\widetilde{\varphi}_\infty$ and $\widetilde{\varphi}_\infty(p) = \widetilde{\varphi}_\infty(q)$ for a couple of points $p, q \in U$.
Choosing $\varepsilon > 0$ smaller than the intrinsic distance between p and q in $\widetilde{M}_\infty$ and such that $B_\varepsilon > 0$, we consider the function

$$L(p,q,t) = |\varphi(p,t) - \varphi(q,t)|/\sqrt{2(T-t)}$$

on $\complement\Omega_\varepsilon \subset M \times M \times [0,T)$. If the minimum of L at time t is lower than $B_\varepsilon > 0$ then it cannot be attained on the boundary of Ω_ε and arguing as in the proof of Proposition 2.2.7, such minimum is nondecreasing. Hence, there is a positive lower bound on

$$\inf_{\complement\Omega_\varepsilon} |\varphi(p,t) - \varphi(q,t)|/\sqrt{2(T-t)}\,.$$

Now we are done, since if we consider two sequences $p_i \to p$ and $q_i \to q$ we have definitely $\widetilde{d}_i(p_i,q_i) > \varepsilon$ and $|\widetilde{\varphi}_i(p_i) - \widetilde{\varphi}_i(q_i)| \to 0$, hence $d_{t_i}(p_i,q_i) > \varepsilon\sqrt{2(T-t_i)}$ which implies that $(p_i,q_i,t_i) \in \complement\Omega_\varepsilon$ and $|\varphi(p_i,t_i) - \varphi(q_i,t_i)|/\sqrt{2(T-t_i)} \to 0$, in contradiction with the previous conclusion. □

Open Problem 3.2.11. The limit hypersurface $\widetilde{M}_\infty$ is unique? That is, independent of the sequence $s_i \to +\infty$?

We have seen in Proposition 1.4.1 that any of these limit hypersurfaces $\widetilde{M}_\infty$ satisfying $\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle = 0$, that we call *homothetic*, generates a homothetically shrinking mean curvature flow given by $M_t = \widetilde{M}_\infty\sqrt{1-2t}$, vanishing at $T = 1/2$.
As we said few explicit examples are available, hyperplanes through the origin, the sphere $\mathbb{S}^n(\sqrt{n})$, the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ and the Angenent's torus we mentioned in Section 1.4, described in [17].

Open Problem 3.2.12. Classify all the complete hypersurfaces (compact or not) satisfying $\mathrm{H} + \langle y \,|\, \nu\rangle = 0$, or at least the ones arising as blow up limits of the flow of a compact and embedded

hypersurface.
This problem is difficult, an equivalent formulation is to find the critical points of the Huisken's functional

$$\int_M e^{-\frac{|y|^2}{2}}\,d\mathcal{H}^n\,.$$

As we will see in the next sections, the classification is possible under the extra hypothesis $\mathrm{H} \geq 0$.

Remark 3.2.13. In the case of a homothetically shrinking hypersurface around a point $x_0 \in \mathbb{R}^{n+1}$ and vanishing at time T, the derivative in the monotonicity formula with the backward heat kernel $\rho_{x_0,T}$ is zero, that is, the integral

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t$$

is constant in time. Conversely, it follows from Proposition 1.4.1 and the uniqueness of the flow, that if such derivative is zero at some time the hypersurface is homothetically shrinking around x_0 as at such time it must satisfy $\mathrm{H} + \frac{\langle x-x_0\,|\,\nu\rangle}{2(T-t)} = 0$ for all its points.

Finally, notice that if the flow φ_t is homothetically shrinking around x_0 for $t \in [0, T)$, the relative rescaled hypersurfaces $\widetilde{\varphi}_s = \frac{\varphi_t - x_0}{\sqrt{2(T-t)}}$ are not moving at all (as a subset of $\mathbb{R}^{n+1}$) and conversely.

We now fix a point $p \in M$ and consider a sequence of rescaled hypersurfaces $\widetilde{\varphi}_{s_i}$, locally smoothly converging (up to reparametrization) to some limit hypersurface $\widetilde{M}_\infty$ which satisfies $\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle = 0$ for every $y \in \widetilde{M}_\infty$.
We want to relate the limit heat density $\Theta(p)$ in Definition 3.2.3 with $\widetilde{M}_\infty$,

$$\begin{aligned}
\Theta(p) &= \lim_{t\to T}\theta(p,t)\\
&= \lim_{i\to\infty}\int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t(s_i))}}}{[4\pi(T-t(s_i))]^{n/2}}\,d\mu_{t(s_i)}\\
&= \lim_{i\to\infty}\int_M \frac{e^{-\frac{|y|^2}{2}}}{(2\pi)^{n/2}}\,d\widetilde{\mu}_{s_i}\\
&= \frac{1}{(2\pi)^{n/2}}\int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n\,,
\end{aligned}$$

where in the last passage we applied Corollary 3.2.8.
In particular, if $\widetilde{M}_\infty$ is a unit multiplicity hyperplane through the origin of $\mathbb{R}^{n+1}$ then $\Theta(p) = \frac{1}{(2\pi)^{n/2}}\int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n = 1$.

Remark 3.2.14. If we choose a time $\tau > 0$ which is strictly less than the maximal time T of existence of the flow and we perform the rescaling procedure around the *nonsingular* point $\widehat{p} = \lim_{t\to\tau}\varphi(p,t) = \varphi(p,\tau)$, being the hypersurface regular around p at time τ, every limit of rescaled hypersurfaces must be flat, actually a union of hyperplanes through the origin. If moreover at $\varphi(p,\tau)$ the hypersurface has no self–intersections, such limit is a *single* hyperplane through the origin and

$$\lim_{t\to\tau}\int_M \frac{e^{-\frac{|x-\varphi(p,\tau)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t = 1\,.$$

This clearly holds for every $p \in M$ if the initial hypersurface is embedded.

Remark 3.2.15. By the previous remark, if $\tau \in (0, T)$ and $x_0 = \varphi_\tau(p)$ we have

$$\lim_{t\to\tau}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t = 1$$

and

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4\tau}}}{[4\pi\tau]^{n/2}}\,d\mu_0 \geq 1$$

by the monotonicity formula, for every $p \in M$.

Lemma 3.2.16 (White [123]). *Among all the smooth, complete, hypersurfaces M in $\mathbb{R}^{n+1}$ satisfying $\mathrm{H} + \langle y\,|\,\nu\rangle = 0$ and $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n < +\infty$, the hyperplanes with unit multiplicity through the origin are the only minimizers of the functional*

$$\frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n\,.$$

Hence, for all such hypersurfaces the value of this integral is at least 1.

Proof. Suppose that there exists a smooth hypersurface $M = M_0$ such that

$$\frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n \leq 1$$

and satisfies $\mathrm{H} + \langle y\,|\,\nu\rangle = 0$, then the flow $M_t = M\sqrt{1-2t}$ is a smooth mean curvature flow in the time interval $(-\infty, 1/2)$.
Choosing a point $y_0 \in \mathbb{R}^{n+1}$ and a time $\tau \leq 1/2$ we consider the limit

$$\lim_{t\to-\infty}\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n\,,$$

where all the integrals are well defined since $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n < +\infty$.
Changing variables, we have

$$\lim_{t\to-\infty}\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n(y) = \lim_{t\to-\infty}\int_{M} \frac{e^{-\frac{|x\sqrt{1-2t}-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)/(1-2t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n(x)\,.$$

As $t \to -\infty$, the sequence of functions inside the integral pointwise converges to the function $e^{-\frac{|x|^2}{2}}/(2\pi)^{n/2}$ and they are definitely uniformly bounded from above, outside some large fixed ball $B_R(0) \subset \mathbb{R}^{n+1}$, by the function $e^{-|x|}$. Since this last function is integrable on M by the hypothesis, using the dominated convergence theorem we get

$$\lim_{t\to-\infty}\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n = \frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|x|^2}{2}}\,d\widetilde{\mathcal{H}}^n \leq 1\,.$$

By the monotonicity formula this implies that

$$\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n \leq 1$$

for every $y_0 \in \mathbb{R}^{n+1}$ and $t < \tau \in (-\infty, 1/2)$.
Choosing now $y_0 \in M$ and $\tau = 0$, repeating the argument in Remark 3.2.14 (in this noncompact case it can be carried on by means of the hypothesis $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n < +\infty$) we have

$$\lim_{t\to 0^-}\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{-4t}}}{[-4\pi t]^{n/2}}\,d\widetilde{\mathcal{H}}^n = 1\,,$$

hence, we conclude that the function

$$\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{-4t}}}{[-4\pi t]^{n/2}}\, d\widetilde{\mathcal{H}}^n$$

is constant equal to 1 for every $t \in (-\infty, 0)$. Even if the evolving hypersurfaces M_t are not compact, by the hypothesis $\int_M e^{-|y|}\, d\widetilde{\mathcal{H}}^n < +\infty$ it is straightforward to check (writing every integral as an integral on M fixed) that the monotonicity formula still holds. Hence, we must have that the right hand side of such formula is identically zero and $\mathrm{H}(y,t) + \frac{\langle y-y_0 \,|\, \nu(y)\rangle}{-2t} = 0$ for every $t < 0$ and $y \in M_t$. Multiplying by $-2t$ and sending t to zero, as $M_t \to M$, we conclude that $\langle y - y_0 \,|\, \nu(y)\rangle = 0$ for every $y, y_0 \in M$. This condition easily implies that M is a hyperplane through the origin of $\mathbb{R}^{n+1}$. □

Remark 3.2.17. It is not known by the author whether the hypothesis $\int_M e^{-|y|}\, d\widetilde{\mathcal{H}}^n < +\infty$ can be removed. Anyway, it is satisfied by every limit hypersurface obtained as blow up limit, by Corollary 3.2.8.

The following corollary is the consequence of Lemma 3.2.16 and the previous discussion about the relation between Θ and the limits of sequences of rescaled hypersurfaces.

Corollary 3.2.18. *The function $\Theta : M \to \mathbb{R}$ satisfies $\Theta \geq 1$ on all M. Moreover, if $\Theta(p) = 1$, every converging sequence of rescaled hypersurfaces $\widetilde{\varphi}_{s_i}$ around $\widehat{p}$ converges to a unit multiplicity hyperplane through the origin of $\mathbb{R}^{n+1}$.*
It follows that $\Sigma \geq 1$ (recall Definition 3.2.3).

Remark 3.2.19. The fact that $\Theta \geq 1$ on all M can also be proved directly using the argument in Remark 3.2.15. Since for every $\tau < T$ we have

$$\lim_{t\to\tau} \int_M \frac{e^{-\frac{|x-\varphi_\tau(p)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\, d\mu_t = 1$$

we get

$$\int_M \frac{e^{-\frac{|x-\varphi_\tau(p)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\, d\mu_t \geq 1$$

for every $t < \tau$. Keeping now $t < T$ fixed and sending $\tau \to T$ we have $\varphi_\tau(p) \to \widehat{p}$ and

$$\theta(p,t) = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t = \lim_{\tau\to T} \int_M \frac{e^{-\frac{|x-\varphi_\tau(p)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\, d\mu_t \geq 1\,.$$

This clearly implies that $\Theta(p) = \lim_{t\to T} \theta(p,t) \geq 1$.

Remark 3.2.20. Rescaling around some $\widehat{p}$, by the discussion after Definition 3.2.2, means rescaling around some *reachable* point. Actually, we could rescale the evolving hypersurfaces around *any* point $x_0 \in \mathbb{R}^{n+1}$ but if $x_0 \notin \mathcal{S}$, as the distance from $\varphi(M,t)$ and x_0 is definitely positive, the limit hypersurface is empty. This would imply that

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t \to 0$$

as $t \to T$.
By Corollary 3.2.18, if instead we consider $x_0 \in \mathcal{S}$, that is, $x_0 = \widehat{p}$ for some $p \in M$, there holds $\Theta(p) \geq 1$. Hence, there is a dichotomy between the points of $\mathbb{R}^{n+1}$, according to the value of this extended limit heat density function which can be either zero or at least one.
Moreover, by looking carefully at the first part of the proof of Lemma 3.2.16 we can see that this fact is independent of the type I hypothesis, it is indeed only a consequence of the upper semicontinuity of $\theta(p,t)$.

Actually, one can say more by the following result of White [123] (see also [35, Theorems 5.6, 5.7] and [115], moreover compare with [20, Theorem 6.11]), which also gives a partial answer to Problem 3.2.11.

Theorem 3.2.21 (White [123, Theorem 3.5])**.** *There exist constants $\varepsilon = \varepsilon(n) > 0$ and $C = C(\varphi_0)$ such that if $\Theta(p) < 1 + \varepsilon$, then $|\mathrm{A}| \leq C$ in a ball of $\mathbb{R}^{n+1}$ around $\widehat{p}$ uniformly in time $t \in [0, T)$.*

If the limit of a subsequence of rescaled hypersurfaces is a hyperplane through the origin, then $\Theta(p) = 1$ and by this theorem there is a ball around $\widehat{p}$ where the curvature is bounded. Then in such a ball, the *unscaled* hypersurfaces φ_t (possibly after a reparametrization) converge locally uniformly in C^0 to some φ_T with uniformly bounded curvature, this implies that the convergence is actually in C^∞ by the interior estimates of Ecker and Huisken in [38]. Hence, it follows easily that the tangent hyperplane to φ_T at the point $\widehat{p}$ coincides with the limit of *any* sequence of rescaled hypersurfaces, that is, there is full convergence and the limit hypersurface is unique, solving affirmatively Problem 3.2.11 in this very special case.

Remark 3.2.22. The strength of White's result is that it does not assume any condition on the sign of H and on the blow up rate of the curvature. The theorem also holds without the type I hypothesis.

Another consequence is that there is a "gap" between the value 1 realized by the hyperplanes through the origin of $\mathbb{R}^{n+1}$ in the functional

$$\frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}} \, d\widetilde{\mathcal{H}}^n$$

and any other smooth, complete hypersurface M in $\mathbb{R}^{n+1}$ satisfying $\mathrm{H} + \langle y \,|\, \nu \rangle = 0$ and $\int_M e^{-|y|} \, d\widetilde{\mathcal{H}}^n < +\infty$.

3.3 Embedded Hypersurfaces with Nonnegative Mean Curvature

In this section we assume that all the hypersurfaces are embedded.

If the compact initial hypersurface is embedded and has $\mathrm{H} \geq 0$ (or at some positive time the evolving hypersurface becomes like that) then the analysis of the previous sections can be pushed forward since we can restrict the class of the possible limits of rescaled hypersurfaces to the ones satisfying these two conditions.

We recall that if $\mathrm{H} \geq 0$ for the initial hypersurface, after a positive time $t_0 > 0$, there exists a constant $\alpha > 0$ such that $\alpha|\mathrm{A}| \leq \mathrm{H} \leq n|\mathrm{A}|$ everywhere on M for every time $t \geq t_0$, by Corollary 2.4.3.
Hence, we can assume in the sequel that for $t \in [0, T)$ we have

$$\frac{\alpha}{\sqrt{2(T-t)}} \leq \max_{p \in M} \mathrm{H}(p, t) \leq \frac{C}{\sqrt{2(T-t)}} \, .$$

Proposition 3.3.1 (Huisken [67, 68], Abresch and Langer [1] in the one–dimensional case)**.** *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, complete, embedded, mean convex hypersurface in $\mathbb{R}^{n+1}$ such that $\mathrm{H} + \langle x \,|\, \nu \rangle = 0$ at every $x \in M$ and there exists a constant C such that $|\mathrm{A}| + |\nabla \mathrm{A}| \leq C$ and $\mathcal{H}^n(M \cap B_R) \leq Ce^R$, for every ball of radius $R > 0$ in $\mathbb{R}^{n+1}$.*
Then, up to a rotation of $\mathbb{R}^{n+1}$, M must be one of only $n + 1$ possible hypersurfaces, namely, either a hyperplane through the origin or the sphere $\mathbb{S}^n(\sqrt{n})$ or one of the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$.

In the special one–dimensional case the only smooth, complete, embedded curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle x \,|\, \nu \rangle = 0$ are the lines through the origin and the unit circle (notice that in this case neither the positivity of the curvature nor the two estimates above are assumed).

Proof. Let us suppose that M is connected. If the theorem is true in this case, it it easy to see that it is not possible to have a nonconnected embedded hypersurface satisfying the hypotheses. Indeed, any connected component has to belong to the list of the statement and every two hypersurfaces in such list either coincide or have some intersections.

We first deal separately with the case $n = 1$.
Fixing a reference point on a curve γ we have an arclength parameter s which gives a unit tangent vector field $\tau = \gamma_s$ and a unit normal vector field ν, which is the counterclockwise rotation of $\pi/2$ in $\mathbb{R}^2$ of the vector τ, then the curvature is given by $k = \langle \partial_s \tau \,|\, \nu \rangle = -\langle \tau \,|\, \partial_s \nu \rangle$.
The relation $k = -\langle \gamma \,|\, \nu \rangle$ implies the ODE for the curvature $k_s = k \langle \gamma \,|\, \tau \rangle$. Suppose that at some point $k = 0$, then also $k_s = 0$ at the same point, hence, by the uniqueness theorem for ODE's we conclude that k is identically zero and we are dealing with a line L which, as $\langle x \,|\, \nu \rangle = 0$ for every $x \in L$, must contain the origin of $\mathbb{R}^2$.
So we suppose that k is always nonzero and possibly reversing the orientation of the curve, we assume that $k > 0$ at every point, that is, the curve is strictly convex.
Computing the derivative of $|\gamma|^2$,

$$\partial_s |\gamma|^2 = 2\langle \gamma \,|\, \tau \rangle = 2k_s/k = 2\partial_s \log k$$

we get $k = Ce^{|\gamma|^2/2}$ for some constant $C > 0$, it follows that k is bounded from below by $C > 0$.
We consider now a new coordinate $\theta = \arccos \langle e_1 \,|\, \nu \rangle$, this can be done for the whole curve as we know that this latter is convex (obviously, as for the arclength parameter s it is only locally continuous, θ "jumps" after a complete round).
Differentiating with respect to the arclength parameter we have $\partial_s \theta = k$ and

$$k_\theta = k_s/k = \langle \gamma \,|\, \tau \rangle \qquad k_{\theta\theta} = \frac{\partial_s k_\theta}{k} = \frac{1 + k\langle \gamma \,|\, \nu \rangle}{k} = \frac{1}{k} - k\,. \tag{3.3.1}$$

Multiplying both sides of the last equation by $2k_\theta$ we get $\partial_\theta [k_\theta^2 + k^2 - \log k^2] = 0$, that is, the quantity $k_\theta^2 + k^2 - \log k^2$ is equal to some constant E along all the curve. Notice that such quantity E cannot be less than 1, moreover, if $E = 1$ we have that k must be constant and equal to 1 along the curve, which consequently must be the unit circle centered at the origin of $\mathbb{R}^2$.
When $E > 1$, it follows that k is uniformly bounded from above, hence recalling that $k = Ce^{|\gamma|^2/2}$, the image of the curve is contained in a ball of $\mathbb{R}^2$ and by the embeddedness and completeness hypotheses, the curve must be closed, simple and strictly convex, as $k > 0$ at every point.
We now suppose that γ is not the unit circle and we look at the critical points of the curvature k. Since $k_{\theta\theta} = \frac{1}{k} - k$, there holds that $k_{\theta\theta} \neq 0$ when $k_\theta = 0$, otherwise this second order ODE for k would imply $k_\theta = 0$ everywhere, hence $k = 1$ identically and we would be in the case of the unit circle. Thus, the critical points of the curvature are not degenerate, hence, by the compactness of the curve they are isolated and finite. Moreover, by looking at the equation for the curvature (3.3.1) we can see easily that $k_{\min} < 1$ and $k_{\max} > 1$.
Suppose now that $k(0) = k_{\max}$ and $k(\overline{\theta})$ is the first subsequent critical value for k, for some $\overline{\theta} > 0$. Then the curvature is strictly decreasing in the interval $[0, \overline{\theta}]$ and again by the second order ODE, the function k (hence also the curve, by integration) is symmetric with respect to $\theta = 0$ and $\theta = \overline{\theta}$. This clearly implies that $k(\overline{\theta})$ must be the minimum $k_{\min}$ of the curvature, as every critical point is not degenerate.

By the four vertex theorem [92, 100], on every closed curve there are at least four critical points of k, as a consequence our curve is composed of at least four pieces like the one described above. Hence, since the curve is closed and embedded the curvature $k(\theta)$ must be a periodic function with period $T > 0$ not larger than π (since 2π is an obvious multiple of the period) and $\overline{\theta} = T/2$. More precisely, the period T must be $2\pi/n$ for some $n \geq 2$.

By a straightforward computation, starting by differentiating the equation $k_{\theta\theta} = \frac{1}{k} - k$, one gets $(k^2)_{\theta\theta\theta} + 4(k^2)_\theta = 4k_\theta/k$, then we compute

$$\begin{aligned}
4\int_0^{T/2} \sin 2\theta \, \frac{k_\theta}{k}\, d\theta &= \int_0^{T/2} \sin 2\theta \, [(k^2)_{\theta\theta\theta} + 4(k^2)_\theta]\, d\theta \\
&= \sin 2\theta (k^2)_{\theta\theta}\big|_0^{T/2} - 2\int_0^{T/2} \cos 2\theta (k^2)_{\theta\theta}\, d\theta + 4\int_0^{T/2} \sin 2\theta (k^2)_\theta\, d\theta \\
&= 2\sin T[k(T/2)k_{\theta\theta}(T/2) + k_\theta^2(T/2)] - 2\cos 2\theta (k^2)_\theta\big|_0^{T/2} \\
&\quad - 4\int_0^{T/2} \sin 2\theta (k^2)_\theta\, d\theta + 4\int_0^{T/2} \sin 2\theta (k^2)_\theta\, d\theta \\
&= 2\sin T[k(T/2)k_{\theta\theta}(T/2) + k_\theta^2(T/2)] \\
&\quad - 4\cos T k(T/2)k_\theta(T/2) + 4k(0)k_\theta(0)\,.
\end{aligned}$$

Now, since $k_\theta(0) = k_\theta(T/2) = 0$ and $k(T/2) = k_{\min}$, using the equation for the curvature $k_{\theta\theta} = 1/k - k$ we get

$$4\int_0^{T/2} \sin 2\theta \, \frac{k_\theta}{k}\, d\theta = 2\sin T(1 - k_{\min}^2)\,,$$

and this last term is nonnegative as $k_{\min} < 1$ and $0 < T \leq \pi$.
Looking at the left hand integral we see instead that the factor $\sin 2\theta$ is always nonnegative, since $T \leq \pi$ and k_θ is always nonpositive in the interval $[0, T/2]$, as we assumed that we were moving from the maximum $k_{\max}$ at $\theta = 0$ to the minimum $k_{\min}$ at $\theta = T/2$ without crossing any other critical point of k. This gives a contradiction so γ must be the unit circle.

We suppose now that $n \geq 2$.
By covariant differentiation of the equation $\mathrm{H} + \langle x \,|\, \nu\rangle = 0$ in an orthonormal frame $\{e_1, \dots, e_n\}$ on M we get

$$\nabla_i \mathrm{H} = \langle x \,|\, e_k\rangle h_{ik}$$

$$\nabla_i\nabla_j \mathrm{H} = h_{ij} + \langle x \,|\, \nu\rangle h_{ik}h_{jk} + \langle x \,|\, e_k\rangle \nabla_i h_{jk} = h_{ij} - \mathrm{H}h_{ik}h_{jk} + \langle x \,|\, e_k\rangle \nabla_k h_{ij} \qquad (3.3.2)$$

where we used Gauss–Weingarten and Codazzi equations (1.1.1), (1.1.3).
Contracting now with g^{ij} and h^{ij} respectively we have

$$\Delta \mathrm{H} = \mathrm{H} - \mathrm{H}|\mathrm{A}|^2 + \langle x \,|\, e_k\rangle \nabla_k \mathrm{H} = \mathrm{H}(1 - |\mathrm{A}|^2) + \langle x \,|\, \nabla \mathrm{H}\rangle \qquad (3.3.3)$$

$$h^{ij}\nabla_i\nabla_j \mathrm{H} = |\mathrm{A}|^2 - \mathrm{tr}(\mathrm{A}^3)\mathrm{H} + \langle x \,|\, e_k\rangle \nabla_k |\mathrm{A}|^2/2$$

which implies, by Simons' identity (1.1.4),

$$\Delta|\mathrm{A}|^2 = 2|\mathrm{A}|^2(1 - |\mathrm{A}|^2) + 2|\nabla \mathrm{A}|^2 + \langle x \,|\, \nabla |\mathrm{A}|^2\rangle\,.$$

From equation (3.3.3) and the strong maximum principle for elliptic equations we see that, since M satisfies $\mathrm{H} \geq 0$ by assumption and $\Delta \mathrm{H} \leq \mathrm{H} + \langle x \,|\, \nabla \mathrm{H}\rangle$, we must either have that $\mathrm{H} \equiv 0$ or $\mathrm{H} > 0$ on all M.
The case $\mathrm{H} \equiv 0$ can be easily handled: as M is complete and x is a tangent vector field on M by the equation $\langle x \,|\, \nu\rangle = 0$, for every point x_0 of M there is a unique solution of the ODE $\gamma'(s) = x(\gamma(s)) = \gamma(s)$ passing through x_0 and contained in M for every $s \in \mathbb{R}$, but such solution is simply the line in $\mathbb{R}^{n+1}$ passing through x_0 and the origin. Thus, M has to be a cone and being smooth the only possibility is a hyperplane through the origin of $\mathbb{R}^{n+1}$.
Therefore we may assume that the mean curvature satisfies the strict inequality $\mathrm{H} > 0$ everywhere (so dividing by H and $|\mathrm{A}|$ is allowed).
Now let $R > 0$ and define η to be the inward unit conormal to $M \cap B_R(0)$ along $\partial\big(M \cap B_R(0)\big)$, which is a smooth boundary for almost every $R > 0$ (by Sard's theorem). Then, supposing that

R belongs to the set $\Omega \subset \mathbb{R}^+$ of the *regular values* of the function $|\cdot|$ restricted to $M \subset \mathbb{R}^{n+1}$, from equation (3.3.3) and the divergence theorem we compute

$$\begin{aligned}
\varepsilon_R &= -\int_{\partial(M\cap B_R(0))} |\mathrm{A}|\langle\nabla\mathrm{H}\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1}(x)\\
&= \int_{M\cap B_R(0)} |\mathrm{A}|\Delta\mathrm{H}e^{-|x|^2/2} + \langle\nabla(|\mathrm{A}|e^{-|x|^2/2})\,|\,\nabla\mathrm{H}\rangle\,d\mathcal{H}^n(x)\\
&= \int_{M\cap B_R(0)} \Big(|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2) + |\mathrm{A}|\langle x\,|\,\nabla\mathrm{H}\rangle\Big)e^{-|x|^2/2}\,d\mathcal{H}^n(x)\\
&\quad+\int_{M\cap B_R(0)} \Big(\frac{1}{2|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\nabla\mathrm{H}\rangle - |\mathrm{A}|\langle x\,|\,\nabla\mathrm{H}\rangle\Big)e^{-|x|^2/2}\,d\mathcal{H}^n(x)\\
&= \int_{M\cap B_R(0)} \Big(|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2) + \frac{1}{2|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\nabla\mathrm{H}\rangle\Big)e^{-|x|^2/2}\,d\mathcal{H}^n(x)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\delta_R &= -\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1}(x)\\
&= \int_{M\cap B_R(0)} \frac{\mathrm{H}}{|\mathrm{A}|}\Delta|\mathrm{A}|^2e^{-|x|^2/2} + \Big\langle\nabla\Big(\frac{\mathrm{H}}{|\mathrm{A}|}|e^{-|x|^2/2}\Big)\,\Big|\,\nabla|\mathrm{A}|^2\Big\rangle\,d\mathcal{H}^n(x)\\
&= \int_{M\cap B_R(0)} \Big(2|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2) + \frac{2\mathrm{H}|\nabla\mathrm{A}|^2}{|\mathrm{A}|} + \frac{\mathrm{H}}{|\mathrm{A}|}\langle x\,|\,\nabla|\mathrm{A}|^2\rangle\Big)e^{-|x|^2/2}\,d\mathcal{H}^n(x)\\
&\quad+\int_{M\cap B_R(0)} \Big(\frac{\langle\nabla\mathrm{H}\,|\,\nabla|\mathrm{A}|^2\rangle}{|\mathrm{A}|} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{2|\mathrm{A}|^3} - \frac{\mathrm{H}}{|\mathrm{A}|}\langle x\,|\,\nabla|\mathrm{A}|^2\rangle\Big)e^{-|x|^2/2}\,d\mathcal{H}^n(x)\\
&= \int_{M\cap B_R(0)} \Big(2|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2) + \frac{2\mathrm{H}|\nabla\mathrm{A}|^2}{|\mathrm{A}|} + \frac{\langle\nabla\mathrm{H}\,|\,\nabla|\mathrm{A}|^2\rangle}{|\mathrm{A}|} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{2|\mathrm{A}|^3}\Big)e^{-|x|^2/2}\,d\mathcal{H}^n(x)\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sigma_R = 2\delta_R - 4\varepsilon_R &= \int_{M\cap B_R(0)} \Big(\frac{4\mathrm{H}|\nabla\mathrm{A}|^2}{|\mathrm{A}|} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{|\mathrm{A}|^3}\Big)\,e^{-|x|^2/2}\,d\mathcal{H}^n(x)\\
&= \int_{M\cap B_R(0)} \Big(4|\mathrm{A}|^2|\nabla\mathrm{A}|^2 - |\nabla|\mathrm{A}|^2|^2\Big)\frac{\mathrm{H}}{|\mathrm{A}|^3}e^{-|x|^2/2}\,d\mathcal{H}^n(x)\,.
\end{aligned}$$

As we have $4|\mathrm{A}|^2|\nabla\mathrm{A}|^2 \geq |\nabla|\mathrm{A}|^2|^2$, this quantity σ_R is nonnegative and nondecreasing in R. If now we show that $\liminf_{R\to+\infty}\sigma_R = 0$ we can conclude that at every point of M

$$4|\mathrm{A}|^2|\nabla\mathrm{A}|^2 = |\nabla|\mathrm{A}|^2|^2\,. \tag{3.3.4}$$

Getting back to the definitions of ε_R and δ_R, we have

$$\begin{aligned}
|\sigma_R| &= \left|-2\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1} + 4\int_{\partial(M\cap B_R(0))} |\mathrm{A}|\langle\nabla\mathrm{H}\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1}\right|\\
&\leq 4e^{-R^2/2}\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}|\nabla|\mathrm{A}|^2| + |\mathrm{A}||\nabla\mathrm{H}|\,d\mathcal{H}^{n-1}\\
&\leq 8e^{-R^2/2}\int_{\partial(M\cap B_R(0))} \mathrm{H}|\nabla\mathrm{A}| + |\mathrm{A}||\nabla\mathrm{H}|\,d\mathcal{H}^{n-1}\\
&\leq Ce^{-R^2/2}\mathcal{H}^{n-1}(\partial(M\cap B_R(0)))\,,
\end{aligned}$$

by the estimates on A and $\nabla\mathrm{A}$ in the hypotheses.
Now, suppose that for every R belonging to the set $\Omega \subset \mathbb{R}^+$ (which is of full measure) and R

larger than some $R_0 > 0$ we have

$$\mathcal{H}^{n-1}(\partial(M \cap B_R(0))) \geq \delta R e^{R^2/4}$$

for some constant $\delta > 0$. Setting x^M to be the projection of the vector x on the tangent space to M, as the function $R \mapsto \mathcal{H}^n(M \cap B_R(0))$ is monotone and continuous from the left and actually continuous at every value $R \in \Omega$, we can differentiate it almost everywhere in $\mathbb{R}^+$ and we have (by the coarea formula, see [41] or [105]),

$$\begin{aligned}
\mathcal{H}^n(M \cap B_R(0)) - \mathcal{H}^n(M \cap B_r(0)) &\geq \int_r^R \frac{d}{d\xi}\mathcal{H}^n(M \cap B_\xi(0))\, d\xi \\
&\geq \int_r^R \int_{\partial(M\cap B_\xi(0))} |\nabla^M |x||^{-1}\, d\mathcal{H}^{n-1}(x)\, d\xi \\
&= \int_r^R \int_{\partial(M\cap B_\xi(0))} |x|/|x^M|\, d\mathcal{H}^{n-1}(x)\, d\xi \\
&\geq \int_r^R \int_{\partial(M\cap B_\xi(0))} d\mathcal{H}^{n-1}(x)\, d\xi\,,
\end{aligned}$$

where the derivative in the integral is taken only at the points where it exists and $\nabla^M|x|$ denotes the projection of the $\mathbb{R}^{n+1}$–gradient of the function $|x|$ on the tangent space to M.
Hence, if $R > r > R_0$ we get

$$\begin{aligned}
\mathcal{H}^n(M \cap B_R(0)) - \mathcal{H}^n(M \cap B_r(0)) &\geq \int_r^R \int_{\partial(M\cap B_\xi(0))} d\mathcal{H}^{n-1}\, d\xi \\
&\geq \delta \int_r^R \xi e^{\xi^2/4}\, d\xi \\
&= 2\delta(e^{R^2/4} - e^{r^2/4})\,,
\end{aligned}$$

then if R goes to $+\infty$, the quantity $\mathcal{H}^n(M \cap B_R(0))e^{-R}$ diverges, in contradiction with the hypotheses of the proposition. Hence, the $\liminf$ of the quantity $e^{-R^2/4}\mathcal{H}^{n-1}(\partial(M \cap B_R(0)))$ as $R \to +\infty$ in the set Ω has to be zero. It follows that the same holds for $|\sigma_R|$ and equation (3.3.4) is proved.

Making explicit such equation, by the equality condition in the Cauchy–Schwartz inequality it immediately follows that at every point there exist constants c_k such that

$$\nabla_k h_{ij} = c_k h_{ij}$$

for every i, j. Contracting this equation with the metric g^{ij} and with h^{ij} we get $\nabla_k \mathrm{H} = c_k \mathrm{H}$ and $\nabla_k |\mathrm{A}|^2 = 2c_k|\mathrm{A}|^2$, hence $\nabla_k \log \mathrm{H} = c_k$ and $\nabla_k \log |\mathrm{A}|^2 = 2c_k$.
This implies that locally $|\mathrm{A}| = \alpha\mathrm{H}$ for some constant $\alpha > 0$ and by connectedness this relation has to hold globally on M.
Suppose now that at a point $|\nabla\mathrm{H}| \neq 0$, then $\nabla_k h_{ij} = c_k h_{ij} = \frac{\nabla_k \mathrm{H}}{\mathrm{H}} h_{ij}$ which is a symmetric 3–tensor by the Codazzi equations (1.1.3), hence $\nabla_k \mathrm{H} h_{ij} = \nabla_j \mathrm{H} h_{ik}$. Computing then in normal coordinates with an orthonormal basis $\{e_1 \ldots, e_n\}$ such that $e_1 = \nabla\mathrm{H}/|\nabla\mathrm{H}|$, we have

$$0 = |\nabla_k \mathrm{H} h_{ij} - \nabla_j \mathrm{H} h_{ik}|^2 = 2|\nabla\mathrm{H}|^2 \left(|\mathrm{A}|^2 - \sum_{i=1}^n h_{1i}^2\right).$$

Hence, $|\mathrm{A}|^2 = \sum_{i=1}^n h_{1i}^2$, that is,

$$h_{11}^2 + 2\sum_{i=2}^n h_{1i}^2 + \sum_{i,j=2}^n h_{ij}^2 = |\mathrm{A}|^2 = h_{11}^2 + \sum_{i=2}^n h_{1i}^2\,,$$

so $h_{ij} = 0$ unless $i = j = 1$, which means that A has rank one.
Thus, we have two possible (not mutually excluding) situations at every point of M, either A has rank one or $\nabla \mathrm{H} = 0$.

If the kernel of A is empty everywhere, A must have rank at least two as we assumed $n \geq 2$, then we have $\nabla \mathrm{H} = 0$ which implies $\nabla \mathrm{A} = 0$ and $h_{ij} = \mathrm{H} h_{ik} h_{kj}$, by equation (3.3.2). This means that all the eigenvalues of A are 0 or $1/\mathrm{H}$. As the kernel is empty $\mathrm{A} = \mathrm{H}g/n$, precisely $\mathrm{H} = \sqrt{n}$ and $\mathrm{A} = g/\sqrt{n}$. Then, the complete hypersurface M has to be the sphere $\mathbb{S}^n(\sqrt{n})$, indeed we compute

$$\Delta^M |x|^2 = 2n + 2\langle x \,|\, \Delta^M x\rangle = 2n + 2\mathrm{H}\langle x \,|\, \nu\rangle = 2n - 2\mathrm{H}^2 = 0\,,$$

by means of the structural equation $\mathrm{H} + \langle x \,|\, \nu\rangle = 0$, hence $|x|^2$ is a harmonic function on M and looking at the point of M of minimum distance from the origin, by the strong maximum principle for elliptic equations, it must be constant on M.

We assume now that the kernel of A is not empty at some point $p \in M$ and let $v_1(p), \dots, v_{n-m}(p) \in T_pM \subset \mathbb{R}^{n+1}$ be a family of unit orthonormal tangent vectors spanning such $(n-m)$–dimensional kernel, that is $h_{ij}(p)v_k^j(p) = 0$. Then, the geodesic $\gamma(s)$ from p in M (which is complete) with initial velocity $v_k(p)$ satisfies

$$\nabla_s(h_{ij}\gamma_s^j) = \mathrm{H}^{-1}\langle \nabla \mathrm{H} \,|\, \gamma_s\rangle h_{ij}\gamma_s^j$$

hence, by Gronwall's lemma there holds $h_{ij}(\gamma(s))\gamma_s^j(s) = 0$ for every $s \in \mathbb{R}$.
Being γ a geodesic in M, the normal to the curve in $\mathbb{R}^{n+1}$ is also the normal to M, then letting κ be the curvature of γ in $\mathbb{R}^{n+1}$, we have

$$\kappa = \left\langle \nu \,\middle|\, \frac{d}{ds}\gamma_s \right\rangle = h_{ij}\gamma_s^i\gamma_s^j = 0\,,$$

thus γ is a straight line in $\mathbb{R}^{n+1}$.
Hence, all the $(n-m)$–dimensional affine subspace $p + S(p) \subset \mathbb{R}^{n+1}$ is contained in M, where we set $S(p) = \langle v_1(p), \dots, v_{n-m}(p)\rangle \subset \mathbb{R}^{n+1}$.
Let now $\sigma(s)$ be a geodesic from p to another point q parametrized by arclength and extend by parallel transport the vectors v_k along σ, then

$$\nabla_s(h_{ij}v_k^j) = \mathrm{H}^{-1}\langle \nabla \mathrm{H} \,|\, \sigma_s\rangle h_{ij}v_k^j$$

and again by Gronwall's lemma it follows that $h_{ij}v_k^j(s) = 0$ for every $s \in \mathbb{R}$, in particular $v_k(q)$ is contained in the kernel of A at $q \in M$. This argument clearly shows that the kernel $S(p)$ of A has constant dimension $n - m$ with $0 < m < n$ (as A is never zero) at every point $p \in M$ and all the affine $(n-m)$–dimensional subspaces $p + S(p) \subset \mathbb{R}^{n+1}$ are contained in M.
Moreover, as $h_{ij}v_k^j = 0$ along the geodesic σ, denoting by $\nabla^{\mathbb{R}^{n+1}}$ the covariant derivative of $\mathbb{R}^{n+1}$ we have

$$\nabla_s^{\mathbb{R}^{n+1}} v_k = \nabla_s v_k + h_{ij}v_k^j\sigma_s^i\nu = 0\,,$$

so the extended vectors v_k are constant in $\mathbb{R}^{n+1}$, which means that the parallel extension is independent of the geodesic σ, that the subspaces $S(p)$ are all a common $(n-m)$–dimensional vector subspace of $\mathbb{R}^{n+1}$ that we denote by S and that $M = M + S \subset \mathbb{R}^{n+1}$.
Since the orthogonal projection map $\pi : M \to S$ is then a submersion, for every vector $y \in S$ we have that $N = M \cap (y + S^\perp)$ is a smooth, complete m–dimensional submanifold of $\mathbb{R}^{n+1}$ and, as $M = M + S$, it is easy to see that $M = N \times S$, which implies that $L = S^\perp \cap M$ is a smooth, complete m–dimensional submanifold of $S^\perp = \mathbb{R}^{m+1}$ with $M = L \times S$.
Moreover, as S is in the tangent space to every point of L, the normal ν to M at a point of L stays in $S^\perp$ so it must coincide with the normal ν^L to L in $S^\perp$, then a simple computation shows that the mean curvature of M at the points of L is equal to the mean curvature H^L of L as a hypersurface of $S^\perp = \mathbb{R}^{m+1}$. This shows that L is a hypersurface in $\mathbb{R}^{m+1}$ satisfying $\mathrm{H}^L + \langle z \,|\, \nu^L\rangle = 0$ for every $z \in \mathbb{R}^{m+1}$. Finally, as by construction the second fundamental form of L has empty kernel, by the previous discussion we have $L = \mathbb{S}^m(\sqrt{m})$ and $M = \mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ and we are done. □

Remark 3.3.2. Notice that it follows that all the possible blow up limits are convex. It is a very important fact, proved by Huisken and Sinestrari [73] (see also White [122]), that the same conclusion also holds at a type II singularity of the flow of a mean convex hypersurface (Theorem 4.2.1 and Proposition 4.2.3 in the next lecture).

Remark 3.3.3. Actually, Abresch and Langer in [1] (and also Epstein and Weinstein in [40]) classified *all* the closed curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle \gamma \,|\, \nu \rangle = 0$ (also the curves with self–intersections). We underline that, even if the techniques are elementary, the proof of such classification result is definitely nontrivial.
The result in the embedded case in these papers is a consequence of the general classification theorem. To the author's knowledge, the "shortcut" presented in the proof above is due to Chou and Zhu [26, Proposition 2.3].

We mention that recently Colding and Minicozzi in [29] proved this classification result assuming only a polynomial volume growth, without any bound on the second fundamental form A.

In dimension $n \geq 2$, without the assumption $\mathrm{H} > 0$ the conclusion is not true, an example is the Angenent's torus in [17]. The following higher dimensional analogue of Abresch and Langer result is an open question.

Open Problem 3.3.4. When $n \geq 2$, is any smooth embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ such that $\mathrm{H} + \langle x \,|\, \nu \rangle = 0$ isometric to a sphere?
This was recently shown by Brendle [21] in the 2–dimensional case.

Then, we have the following conclusion.

Theorem 3.3.5. *Let the compact, initial hypersurface be embedded and with $\mathrm{H} \geq 0$. Then, at a type I singularity, every limit hypersurface obtained by rescaling around a reachable point, up to a rotation in $\mathbb{R}^{n+1}$, must be either a hyperplane for the origin, the sphere $\mathbb{S}^n(\sqrt{n})$ or one of the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$.*

We discuss now what are the possible values of the limit heat density function, following Stone [115]. As $\Theta(p)$ is the value of the Huisken's functional on any limit of rescaled hypersurfaces and since these latter are "finite", we have that the possible values of $\Theta(p)$ are 1 in the case of a hyperplane and

$$\Theta^{n,m} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^n$$

for $m \in \{1, \ldots, n\}$.
A straightforward computation gives for $m > 0$

$$\Theta^{n,m} = \left(\frac{m}{2\pi e}\right)^{m/2} \omega_m$$

where ω_m denotes the volume of the unit m–sphere.
Notice that $\Theta^{n,m}$ does not depend on n so we can simply write $\Theta^m = \Theta^{n,m}$.

Lemma 3.3.6 (Stone [115])**.** *The values of Θ^m are all distinct and larger than 1 for $m > 0$. Indeed the numbers $\{\Theta^m \,|\, m = 1, 2, \ldots\}$ form a strictly decreasing sequence in $m \in \mathbb{N}$, with $\Theta^m \searrow \sqrt{2}$ as $m \to \infty$.*

By all this discussion we conclude that the "shape" of the limit hypersurfaces arising from a blow up at a type I singularity of mean curvature flow of a compact, embedded, mean convex hypersurface, is classified by the value of the limit heat density function at the blow up point. Being all these values distinct, such a "shape" is actually independent of the chosen sequence of times for the blow up procedure.

Hence, we can summarize what happens as $t \to T$ as follows: first, we see that it is not possible that at every point $p \in M$ we always get an hyperplane as limit of rescaling, since it would imply that $\Theta(p) = 1$ for every $p \in M$ (hence $\Sigma = 1$), and White's Theorem 3.2.21 would

imply that the curvature is uniformly bounded in a ball around any reachable point $\widehat{p}$. Being such subset of $\mathbb{R}^{n+1}$ compact, it would follow that the curvature is uniformly bounded as $t \to T$, which is a contradiction.
In the other cases, we get as a blow up limit a "cylinder" or a sphere. This latter case is particular, since (by the smooth convergence of the rescaled sequence to the limit hypersurface) it implies that the evolving hypersurface has become convex at some time $t < T$. Then, its flow is described by the following pair of theorems.

Theorem 3.3.7 (Gage and Hamilton [46, 47, 48])**.** *Under the curvature flow a convex closed curve in $\mathbb{R}^2$ smoothly shrinks to a point in finite time. Rescaling in order to keep the length constant, it converges to a circle in C^∞.*

Theorem 3.3.8 (Huisken [65])**.** *Under the mean curvature flow a compact and convex hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 2$ smoothly shrinks to a point in finite time. Rescaling in order to keep the* Area *constant, it converges to a sphere in C^∞.*

Remark 3.3.9. The theorem for curves is not merely a consequence of the general result. The proof in dimension $n \geq 2$ does not work in the one–dimensional case.

Actually, the C^∞–convergence to a circle or to a sphere is exponential.

At the end of Section 4.1 of the next lecture, we will show a line of proof of Theorem 3.3.8 by Hamilton in [60], different from the original one. Another proof was also given by Andrews in [9], analyzing the behavior of the eigenvalues of the second fundamental form close to the singular time.

Notice that in the case of an evolving surface, at a type I singularity we conclude that either it becomes convex or rescaling around some reachable point $\widehat{p}$ we get as a blow up limit a cylinder, since in dimension 2 the only possibilities are a plane, a sphere or a cylinder.

3.4 Embedded Closed Curves in the Plane

The case of an embedded, closed curve γ in $\mathbb{R}^2$ is special, indeed the classification theorem 3.3.1 holds without *a priori* assumptions on the curvature. So there are only two possible limits of rescaled curves without self–intersections, either a line through the origin or the circle $\mathbb{S}^1$. As we already said, in this very special case Problem 3.2.11 is solved affirmatively, the limit is always unique. Arguing as in the previous section, we then have the following conclusion.

Theorem 3.4.1. *Let $\gamma \subset \mathbb{R}^2$ be a simple closed curve, then every curve obtained by limit of rescalings around a type I singular point of its motion by curvature is the circle $\mathbb{S}^1$.*
As a consequence, if a simple closed curve is developing a type I singularity, at some time the curve becomes convex and it shrinks to a point getting asymptotically circular at the singular time.

We mention here that an extensive and deep analysis of the behavior of general curves moving by curvature (even when the ambient is a generic surface different from $\mathbb{R}^2$) is provided by the pair of papers by Angenent [14, 16] (see also the discussion in [17]).