Lecture 2: Isoperimetric methods for the curve-shortening flow and for the Ricci flow on surfaces

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Main idea:
Try to apply a maximum principle to control an isoperimetric profile, in two situations:

• For a metric on the 2-sphere evolving by Ricci flow; and

• For a region enclosed by a simple closed curve in the plane evolving by curve shortening flow.

Although the situation is easier to visualise for curves, the computations work out more nicely for the Ricci flow setting, so I will begin by explaining what happens in that case, and later come back to the curve shortening flow.
There are close parallels between the two situations: In both cases we show that a 'maximum principle' argument can be applied to a suitably defined isoperimetric profile, keeping it above a time-dependent barrier satisfying a certain parabolic differential inequality. By explicitly constructing solutions of this differential inequality we get unexpectedly strong results, including explicit sharp exponential rates of convergence and immediate control on the curvature.

Given an initial metric g_0 on a compact manifold M, Ricci flow produces a smoothly varying family of metrics g_t moving in the direction of the Ricci tensor:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

This was introduced by Hamilton in 1982 in the 3D setting.

Basic facts:

• A solution always exists for a short time.

• The solution continues to exist and remains smooth as long as the curvature remains bounded.

In the 2D setting the curvature tensor is entirely determined by the Gauss curvature K, and the equation simplifies to

$$\frac{\partial}{\partial t} g_{ij} = -2K g_{ij}.$$

In particular the metric remains in the same conformal class as it evolves. It is convenient to rescale the metrics to have fixed area 4π, and readjust the time parameter accordingly, giving the modified flow

$$\frac{\partial}{\partial t} g_{ij} = -(2K - \chi) g_{ij}$$

where χ is the Euler characteristic of the surface.

Despite the simpler geometry, the Ricci flow on surfaces is in some ways more difficult to handle than the 3D Ricci flow. The main result was proved finally by Bennett Chow in 1991, who completed the last remaining case in a program mostly carried out by Hamilton:

Theorem (Hamilton, Chow): For any metric g_0 on a compact surface, the solution of the normalised Ricci flow exists for all $t \geq 0$ and converges smoothly to a metric of constant curvature $K = \frac{\chi}{2}$ as $t \to \infty$

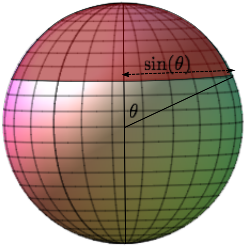
The most difficult case, proved by Chow, is for an arbitrary (not necessarily positively curved) metric on S^2, and it is in this setting that the isoperimetric profile methods apply most directly.

Given a metric g of area 4π on S^2, the isoperimetric profile is the function on $[0,4\pi]$ defined by

$$\mathcal{I}_g(a) = \inf\{|\partial\Omega|_g : \Omega \subset S^2, |\Omega|_g = a\}.$$

Example: The isoperimetric profile of the standard metric $\bar{g}$ on S^2 is given by

$$\mathcal{I}_{\bar{g}}(a) = \sqrt{4\pi a - a^2}.$$



Isoperimetric regions are spherical caps.

$$L = 2\pi \sin\theta;$$

$$A = \int_0^\theta 2\pi \sin\phi \, d\phi = 2\pi(1-\cos\theta).$$

$$L^2 = 4\pi^2 \sin^2\theta = 4\pi^2(1-\cos^2\theta) = A(4\pi - A).$$

Remark: For any metric, we have $\mathcal{I}(a) \simeq \sqrt{4\pi a}$ as $a \to 0$.

The argument I am going to describe builds on an earlier one by Hamilton:

Theorem (Hamilton 1995): If a metric g_0 on S^2 has $\mathcal{I}_{g_0} \geq c\mathcal{I}_{\bar{g}}$ for some $c > 0$, then this remains true under Ricci flow for all $t \geq 0$.

Start by asking the following question:

Consider (smooth, positive) functions $\varphi(a,t)$. Suppose we know that $\mathcal{I}_{g_0}(a) \geq \varphi(a,0)$ for all a. What do we need to assume about φ to ensure that $\mathcal{I}_{g_t}(a) \geq \varphi(a,t)$ for all a and all $t \geq 0$ under Ricci flow?

To answer this we first make some simplifying assumptions: Assume that $\lim_{a\to 0} \frac{\varphi(a,t)}{\sqrt{a}} < 4\pi$ for each $t \geq 0$, and that $\mathcal{I}_{g_0} > \varphi$ for $a \in (0,4\pi)$.

Suppose that the inequality $\mathcal{I} > \varphi$ eventually fails. By the assumption, it follows that there exists $\bar{t} > 0$ and $\bar{a} \in (0,4\pi)$ such that $\mathcal{I}_{g_t}(a) \geq \varphi(a,t)$ for all $0 \leq t \leq \bar{t}$ and $a \in (0,4\pi)$, and $\mathcal{I}_{g_{\bar{t}}}(\bar{a}) = \varphi(\bar{a},\bar{t})$.

It follows that there exists a region $\Omega_0 \subset S^2$ with smooth boundary, $|\Omega_0|_{g_{\bar{t}}} = \bar{a}$, and

$$|\partial\Omega|_{g_{\bar{t}}} = \varphi(\bar{a},\bar{t}).$$

But we also have

$$|\partial\Omega|_{g_t} \geq \varphi(|\Omega|_{g_t},t)$$

for $0 < t \leq \bar{t}$, so that

$$\frac{\partial}{\partial t}\left(|\partial\Omega| - \varphi(|\Omega|,t)\right)\Big|_{t=\bar{t}} \leq 0.$$

Since $\mathcal{I} \geq \varphi$ at $t = \bar{t}$, we also know: If Ω_s is any smoothly varying family of regions deforming Ω_0, then $|\partial\Omega_s|_{g_{\bar{t}}} \geq \varphi(|\Omega_s|_{g_{\bar{t}}}, \bar{t})$ for all s, with equality at $s = 0$. The first derivative therefore vanishes:

$$\frac{\partial}{\partial s}\left(|\partial\Omega_s| - \varphi(|\Omega_s|, \bar{t})\right)\Big|_{s=0} = 0.$$

The second derivative is non-negative:

$$\frac{\partial^2}{\partial s^2}\left(|\partial\Omega_s| - \varphi(|\Omega_s|, \bar{t})\right)\Big|_{s=0} \geq 0.$$

We will compute these identities and ask when they contradict each other. Under a variation of Ω in which the boundary moves in the normal direction with speed f, the first variation is computed as follows:

$$\partial_s |\Omega| = \int_{\partial\Omega} f \, ds,$$

while

$$\partial_s |\partial\Omega| = \int_{\partial\Omega} kf \, ds,$$

where k is the geodesic curvature of $\partial\Omega$. This gives

$$0 = \partial_s | \left(|\partial\Omega| - \varphi(|\Omega|)\right) = \int_{\partial\Omega} kf \, ds - \varphi' \int_{\partial\Omega} f \, ds = \int_{\partial\Omega} f(k - \varphi') \, ds.$$

Since f is arbitrary, we conclude that $k = \varphi'(\bar{a}, \bar{t})$ everywhere on $\partial\Omega_0$.

Next we compute the second variation. In fact we only need this for a specific variation: $f = 1$. From above we have

$$\frac{\partial}{\partial s}\left(|\partial\Omega_s| - \varphi(|\Omega_s|)\right) = \int_{\partial\Omega} k ds - \varphi'|\partial\Omega_s| = 2\pi\chi(\Omega) - \int_{\Omega} K\,d\mu - \varphi'|\partial\Omega_s|,$$

where the last step is Gauss-Bonnet. Differentiating again, we find

$$\begin{aligned}\frac{\partial^2}{\partial s^2}\left(|\partial\Omega_s| - \varphi(|\Omega_s|)\right) &= -\int_{\partial\Omega} K\,ds - \varphi'\int_{\partial\Omega} k\,ds - \varphi''|\partial\Omega_s|^2 \\ &= -\int_{\partial\Omega} K\,ds - (\varphi')^2\varphi - \varphi''\varphi^2 \geq 0.\end{aligned}$$

Finally, the time derivative: Since $\partial_t g = -2(K-1)g$, we have

$$\frac{\partial}{\partial t}|\partial\Omega| = \frac{\partial}{\partial t}\int_{\partial\Omega}\sqrt{g(\partial_u,\partial_u)}du = -\int_{\partial\Omega}(K-1)\,ds = \varphi - \int_{\partial\Omega} K\,ds.$$

Similarly we have

$$\frac{\partial}{\partial t}|\Omega| = -2\int_{\Omega}(K-1)\,d\mu = 2|\Omega| - 2\int_{\Omega} K = 2\bar{a} - 2\left(2\pi\chi(\Omega) - \int_{\partial\Omega} k\,ds\right),$$

where the last step is Gauss-Bonnet.

Putting these together we find:

$$\frac{\partial}{\partial t}(|\partial\Omega| - \varphi(|\Omega|,t)) = \varphi - \int_{\partial\Omega} K\,ds - \varphi'\left(2\bar{a} - 4\pi\chi(\Omega) + 2\varphi'\varphi\right) - \frac{\partial\varphi}{\partial t} \leq 0.$$

We now have two inequalities for $\int_{\partial\Omega} K\,ds$:

$$-(\varphi')^2\varphi - \varphi''\varphi^2 \geq \int_{\partial\Omega} K\,ds \geq \varphi - 2\bar{a}\varphi' + 4\pi\chi(\Omega)\varphi' - 2(\varphi')^2\varphi - \frac{\partial\varphi}{\partial t}$$

So we have a contradiction, provided the right-hand side is strictly greater than the left:

$$\frac{\partial\varphi}{\partial t} < \varphi^2\varphi'' - \varphi(\varphi')^2 + \varphi - 2\bar{a}\varphi' + 4\pi\chi(\Omega)\varphi'.$$

To finish the picture we need one more piece of information:

Claim: If φ is strictly concave, then $\chi(\Omega) = 1$.

Proof: It is enough to show that both Ω and $S^2 \setminus \Omega$ are connected. If Ω is not connected, then write $\Omega = \Omega_1 \cup \Omega_2$ and $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. But then

$$\varphi(|\Omega_1| + |\Omega_2|) = \varphi(|\Omega|) = |\partial\Omega| = |\partial\Omega_1| + |\partial\Omega_2| \geq \varphi(|\Omega_1|) + \varphi(|\Omega_2|) > \varphi(|\Omega_1| + |\Omega_2|)$$

by strict concavity, yielding a contradiction. The argument for the complement is identical.

Theorem

Suppose $\varphi : [0,4\pi] \times [0,\infty)$ *is smooth, reflection-symmetric, and strictly concave for* $0 < a < 4\pi$*, and is a solution of the differential inequality*

$$\frac{\partial \varphi}{\partial t} \leq \varphi^2 \varphi'' - \varphi(\varphi')^2 + \varphi - 2\bar{a}\varphi' + 4\pi\varphi',$$

with $\lim_{a\to 0} \frac{\varphi(a,t)}{\sqrt{a}} \leq 4\pi$ *for each* $t \geq 0$*. If* g_0 *satisfies* $\mathcal{I}_{g_0} \geq \varphi(.,0)$ *and* (g_t) *evolves by the normalised Ricci flow, then* $\mathcal{I}_{g_t} \geq \varphi(.,t)$ *for each* $t \geq 0$*.*

Proof: Check that $(1-\varepsilon)\varphi$ satisfies all the strict inequalities we imposed before. □

Now the punchline: How do we find a solution of the differential inequality?

Theorem

Let $\tilde{g}_0$ *be an axially symmetric metric on* S^2 *with north-south reflection symmetry, with* K *positive and increasing from the equator to the poles. Let* $\tilde{g}_t$ *be the solution of normalised Ricci flow with this initial data. Then* $\varphi(a,t) := \mathcal{I}_{\tilde{g}_t}(a)$ *satisfies the inequalities required in the previous theorem, with equality holding throughout.*

Proof: Ritoré proved that the isoperimetric regions are the spherical caps bounded by lines of latitude. Carry through the computation with $\tilde{g}$ and φ. Equality holds for all a and t, so the time variation and second variation inequalities are equalities. □

There is a remarkable explicit solution to the Ricci flow on a 2-sphere, found by King and later independently by Rosenau. Here the metric is rotationally symmetric, and is given explicitly as follows, as a metric on $\mathbb{R} \times [0, 4\pi]$ for $t \in \mathbb{R}$:

$$g = u(x,t)(dx^2 + dy^2),$$

where

$$u(x,t) = \frac{\sinh(e^{-2t})}{2e^{-2t}(\cosh(x) + \cosh(e^{-2t}))}.$$

This satisfies all the requirements of the previous theorem, and we can compute the corresponding solution φ of the required differential equation.

Important: As $t \to -\infty$, the isoperimetric profile φ approaches zero. This ensures that for any metric g_0 on S^2, we have $\mathcal{I}_{g_0} \geq \varphi(.,t_0)$ for sufficiently negative t_0. The theorem then gives $\mathcal{I}_{g_t} \geq \varphi(.,t_0 + t)$ for all $t \geq 0$.

What can we deduce from the bound on the isoperimetric profile? Any lower bound gives large-scale information about the shape: Solutions cannot become very long and thin. The comparison we have just proved is much stronger:

Theorem (Curvature bound)

In the situation of the previous theorem, we have $\max_{S^2} K(.,t) \leq \max_{S^2} \tilde{K}(.,t)$, *where* $\tilde{K}$ *is the Gauss curvature of the metric* $\tilde{g}$.

Proof: A more careful analysis of the isoperimetric profile for small a (obtained by computing the lengths and areas of small geodesic balls) gives

$$\mathcal{I}_g(a) = \sqrt{4\pi a} - \frac{\sup_{S^2} K}{4\sqrt{pi}} a^{3/2} + O(a^2) \qquad \text{as } a \to 0.$$

This holds for both g_t and $\tilde{g}_t$ and the result follows since

$$\mathcal{I}_{g_t} \geq \mathcal{I}_{\tilde{g}_t}. \quad \square$$

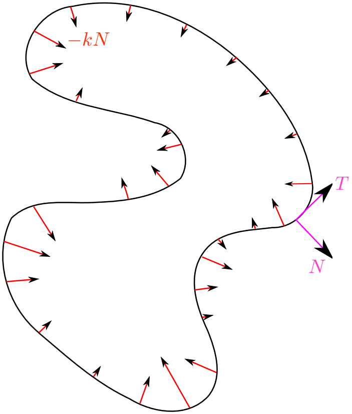
Corollary: For any solution of normalized Ricci flow on S^2, there exists t_0 such that

$$K(x,t) \leq \coth\left(e^{-2(t+t_0)}\right) e^{-2(t+t_0)} \leq 1 + \frac{1}{2} e^{-4(t+t_0)}.$$

From here the convergence of the flow is very easy.

In the curve shortening flow, a curve in the plane moves with speed equal to curvature in the normal direction:

$$\frac{\partial X}{\partial t} = kN.$$



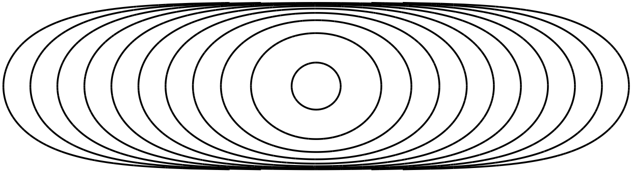
Basic facts:

- Solutions exist for a short time
- Disjoint curves remain disjoint
- Embedded solutions remain embedded
- A singularity must occur in finite time
- The final time is characterised by blowup of the curvature: If curvature stays bounded, the curve stays smooth.

The curve can be normalized to fix the enclosed area π ('Normalised CSF')

$$\frac{\partial X}{\partial t} = -kN + X.$$

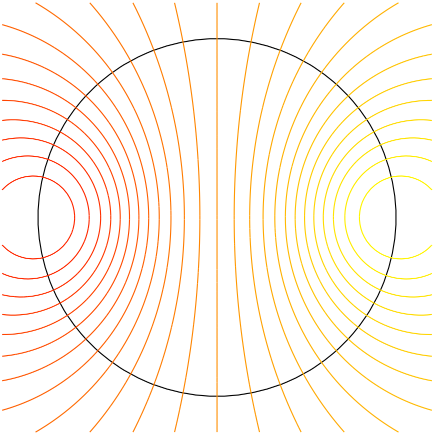
Special solutions: A circle is stationary under NCSF.
The grim reaper $y = \log\cos x + t$ translates vertically under CSF.
The 'paperclip' or 'Angenent oval' is an exact solution of CSF given by $e^t \cos y = \cosh x$ for $t < 0$.



Theorem (Grayson, 1986)

If the initial curve is closed and embedded (with area π), the solution of NCSF exists for all $t \geq 0$ and converges smoothly to the unit circle.

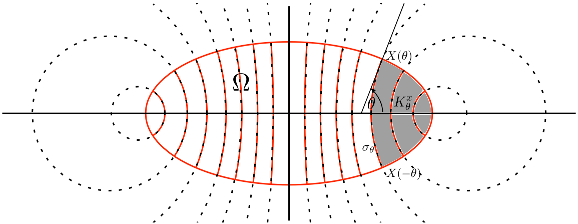
Isoperimetric argument: Let Ω_t be the region enclosed by the curve at time t.
Define $\mathcal{I}(a) = \inf\{|\partial_{\Omega_t} A| : A \subset \Omega, |A| = a\}$.
Example: The circle



- As before, ask when the inequality $\mathcal{I}_{X_t} \geq \varphi(.,t)$ is preserved under NCSF.
- By requiring that time variation and second spatial variation inequalities produce a contradiction, deduce a differential inequality for φ:

$$\frac{\partial \varphi}{\partial t} \leq \mathcal{F}(\varphi'', \varphi', \varphi, a).$$

- Observe that equality holds if φ is constructed from the isoperimetric profile of a convex curve which is symmetric in both axes and has only four vertices.



- Use the paperclip to produce an explicit comparison.
- As before the isoperimetric bound gives a curvature estimate:

$$k(x,t) \leq 1 + Ce^{-2t}.$$

- Long time existence and convergence follow.