Lecture 1: Moduli of continuity, eigenvalues and the fundamental gap conjecture

Ben Andrews

Mathematical Sciences Institute, Australian National University

Winter School of Geometric Evolution Equations
Regensburg, Feb 16–19, 2016

The first three lectures share a common theme: Understanding the behaviour of parabolic or elliptic equations by applying the maximum principle to ‘non-local’ functions (that is, functions involving several points or the global properties of the solution in some way).

I will explore this over the three talks in three quite different applications, related by the similarity of the techniques used.

First lecture: Applying maximum principles to control the modulus of continuity and related properties of solutions of the heat equation. The main application of these ideas is to estimating eigenvalues of the Laplacian. Much of this describes joint work with Julie Clutterbuck (Monash).

Second lecture: Applying maximum principles to estimate the isoperimetric profile of an evolving geometric object (joint work with Paul Bryan, now at Warwick). This produces surprisingly strong results with very little effort, and gives easy proofs of two of the more famous results in the area:

• Grayson's theorem on shrinking embedded closed curves to 'round points' under curve shortening flow; and

• Hamilton and Chow's results about deforming metrics on surfaces to constant curvature under Ricci flow.

Third lecture: Applying some related ideas to the mean curvature flow in higher dimensions. This gives a 'non-collapsing' estimate for hypersurfaces moving by mean curvature, which is the basis for a powerful new regularity theory for mean-convex hypersurfaces moving by mean curvature flow, and when adapted to the elliptic setting (as was done by Brendle) yields a proof of the Lawson conjecture on minimal surfaces in the 3-sphere.

Fourth lecture: This will depart from the theme of the other talks, and consider the evolution of hypersurfaces by Gauss curvature. I will discuss some recent progress on this old problem, which mostly uses variational arguments.

- Controlling modulus of continuity for solutions of the classical heat equation

- Eigenvalue inequality 1: The Payne-Weinberger inequality

- Modulus of continuity for heat equations on manifolds

- Eigenvalue inequality 2: Sharp lower bounds involving Ricci curvature and diameter

- Sharp Log-concavity of the first Dirichlet eigenfunction

- and the fundamental gap conjecture

The 1D heat equation and Kruzhkov's method

To motivate what we do later, I want to mention a nice observation due to Kruzhkov in the 1960s, which is very useful in dealing with parabolic equations in one spatial variable. Let u be a solution of the 1D heat equation $u_t = u''$. Then consider the function

$$v(x,y,t) = u(y,t) - u(x,t).$$

Observe that $v(x,x,t) = 0$ for every x and t, and we have

$$v_t(x,y,t) = u''(y,t) - u''(x,t) = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = \Delta v.$$

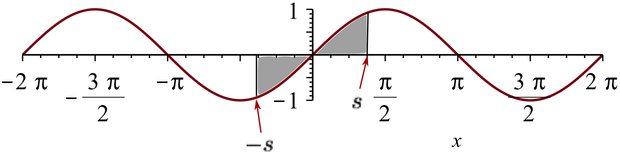
So v solves the heat equation with zero Dirichlet boundary condition on $\{(x,y,t) : \; y > x, \; t \geq 0\}$.

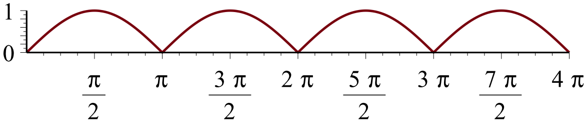
If u is bounded, then so is v and we can use a simple barrier argument to prove a boundary gradient estimate for v. Since $Dv(x,x,t) = (u'(x,t), -u'(x,t))$, the boundary gradient for v implies a global gradient estimate for u.

Consider a solution $u(x,t)$ of the heat equation in one spatial variable. For simplicity assume that u is spatially periodic, so that $u(x+L,t) = u(x,t)$ for all x and t, for some $L > 0$. We define the *modulus of continuity* to be

$$\omega(s,t) = \sup\left\{ \frac{u(y,t)-u(x,t)}{2} \;\middle|\; \frac{|y-x|}{2} = s \right\}.$$

Example: If $u(x) = \sin(x)$ then $\omega(s) = |\sin(s)|$.





Note ω is generally non-smooth.

More generally: If u is odd, $L/2$-antiperiodic, and concave on $(0, L/2)$, then $\omega(s) = |u(s)|$ for all s. Evolving such a function by the heat equation preserves all these properties, so we then have $\omega(s,t) = |u(s,t)|$ for all s and t. In particular, the modulus of continuity ω evolves by the heat equation (on the interval $[0, L/2]$).

Claim: If u is any L-periodic solution of the heat equation, then ω is a viscosity subsolution of the heat equation with Dirichlet boundary conditions on $[0,L]\times[0,\infty)$.

Proof: We have to show the following: If φ is a smooth function defined on a neighbourhood of $(s_0,t_0)\in(0,L)\times(0,t_0]$ such that $\varphi\geq\omega$ with equality at (s_0,t_0), then $(\partial_t-\partial_x^2)\varphi\big|_{(s_0,t_0)}\leq 0$.

We have $\varphi(s,t)\geq\omega(s,t)\geq\frac{u(y,t)-u(x,t)}{2}$ for any x,y with $\frac{y-x}{2}=s$, and equality holds in the first inequality at (s_0,t_0). Also, there exist points x_0 and y_0 with $\frac{y_0-x_0}{2}=s_0$ such that equality holds in the second inequality. It follows that $Z(x,y,t)=u(y)-u(x)-2\varphi(\frac{y-x}{2})$ is non-positive for $t\leq t_0$ and zero at (x_0,y_0,t_0), so

$$0\leq\frac{\partial Z}{\partial t}\bigg|_{(x_0,y_0,t_0)}=u''(y_0,t_0)-u''(x_0,t_0)-2\frac{\partial\varphi}{\partial t}(s_0,t_0),$$

while

$$0=DZ\Big|_{(x_0,y_0,t_0)}=\begin{bmatrix}u'(y_0,t_0)-\varphi'(s_0,t_0)\\ -u'(x_0,t_0)+\varphi'(s_0,t_0)\end{bmatrix},$$

and

$$0\geq D^2Z\Big|_{(x_0,y_0,t_0)}=\begin{bmatrix}u''(y_0,t_0)-\frac{\varphi''(s_0,t_0)}{2} & \frac{\varphi''(s_0,t_0)}{2}\\ \frac{\varphi''(s_0,t_0)}{2} & -u''(x_0,t_0)-\frac{\varphi''(s_0,t_0)}{2}\end{bmatrix}.$$

$$0 \geq D^2 Z\Big|_{(x_0,y_0,t_0)} = \begin{bmatrix} u''(y_0,t_0) - \frac{\varphi''(s_0,t_0)}{2} & \frac{\varphi''(s_0,t_0)}{2} \\ \frac{\varphi''(s_0,t_0)}{2} & -u''(x_0,t_0) - \frac{\varphi''(s_0,t_0)}{2} \end{bmatrix}.$$

To make best use of the latter we compute

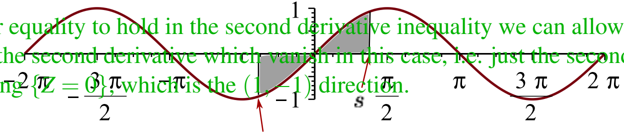
$$0 \geq D^2 Z\Big|_{(x_0,y_0,t_0)}((1,-1),(1,-1)) = u''(y_0,t_0) - u''(x_0,t_0) - 2\varphi''(s_0,t_0).$$

Subtracting this from the time derivative inequality gives

$$0 \leq \left(\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)^2\right) Z\Big|_{(x_0,y_0,t_0)} = 2\left(\varphi'' - \frac{\partial \varphi}{\partial t}\right)\Big|_{(s_0,t_0)}$$

as required. □

Remark: Why didn't we compute $\left(\frac{\partial}{\partial t} - \Delta\right) Z$? This only gives $u''(y_0,t_0) - u''(x_0,t_0) - \varphi''(s_0,t_0)$, and we would deduce only that $\frac{\partial \varphi}{\partial t} \leq \frac{1}{2}\varphi''$. Why is the particular component of the second derivative we used the right choice?
The answer can be seen by considering the case where we know $\omega = u$ on $(0,L)$. Then taking $\varphi = u$, we have $Z = 0$ on the set $\{x + y = 0\}$.



For equality to hold in the second derivative inequality we can allow only those parts of the second derivative which vanish in this case, i.e. just the second derivative along $\{Z = 0\}$, which is the $(1,-1)$ direction.

Now consider the heat equation in higher dimensions, and let us also deal with some natural boundary conditions. For this argument the most natural boundary condition is the Neumann condition, so we let Ω be a (smoothly) bounded convex domain in $\mathbb{R}^n$, take ν to be the outward unit normal, and consider solutions to the Neumann heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{on } \Omega; \\ D_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

We define the modulus of continuity by

$$\omega(s,t) := \sup\left\{ \frac{u(y,t)-u(x,t)}{2} \;\middle|\; x,y \in \Omega,\ \frac{|y-x|}{2} = s \right\}.$$

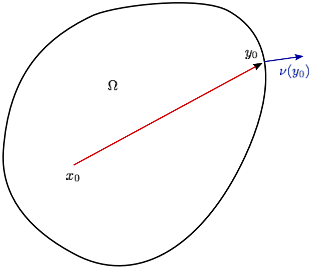
This is defined on $[0, D/2] \times [0, \infty)$, where $D = \mathrm{diam}(\Omega) = \sup\{|y-x| : x,y \in \Omega\}$. Note that we again have a particular situation where ω satisfies the heat equation: Suppose Ω is a cylinder $A \times [-a/2, a/2]$ (where A is a convex subset of $\mathbb{R}^{n-1}$) and suppose $u(x,y,t)$ is a function of y and t only: $u(x,y,t) = f(y,t)$. Then u satisfies the Neumann heat equation on Ω if and only if f satisfies the one-dimensional Neumann heat equation on $[-a/2, a/2]$. In this case, provided f is odd, increasing, and concave for $y > 0$, we have $\omega(s,t) = f(s,t)$ (for $0 < s < a$) and so ω satisfies the one-dimensional heat equation.

Claim: If u is any solution of the Neumann heat equation on Ω, then ω is a viscosity subsolution of the heat equation on $[0, D/2] \times [0, \infty)$ with $\omega(0,t) = 0$ and $\omega'(D/2, t) = 0$ (with respect to non-decreasing barriers).

Proof: We need to consider a non-decreasing smooth function φ touching ω from above at (s_0, t_0). Exactly as before we have for all $x, y \in \Omega$ and all $t \in [0, t_0]$ that

$$Z(x,y,t) := u(y,t) - u(x,t) - 2\varphi\left(\frac{|y-x|}{2}, t\right) \le 2\omega\left(\frac{|y-x|}{2}, t\right) - 2\varphi\left(\frac{|y-x|}{2}, t\right) \le 0,$$

with equality at (x_0, y_0, t_0), where $\frac{|y_0 - x_0|}{2} = s_0$ and x_0 and y_0 attain equality in the definition of $\omega(s_0, t_0)$.
We first consider the possibility that y_0 is in the boundary of Ω:



Then

$$\begin{aligned} 0 &\le DZ\Big|_{(x_0,y_0,t_0)}(0, \nu(y_0)) \\ &= D_\nu u(y_0) - \varphi' \frac{y_0 - x_0}{|y_0 - x_0|} \cdot \nu(y_0) \\ &\le 0, \end{aligned}$$

so equality holds throughout.

The case where x_0 is in the boundary of Ω is similar.

It follows that whether or not x_0 or y_0 is in the boundary, we have

$$\left.\frac{\partial Z}{\partial t}\right|_{(x_0,y_0,t_0)} \geq 0;$$

and

$$D^2 Z\big|_{(x_0,y_0,t_0)} \leq 0.$$

The first gives

$$\Delta u(y_0,t_0) - \Delta u(x_0,t_0) - 2\frac{\partial \varphi}{\partial t} \geq 0.$$

We only compute the components of $D^2 Z$ which produce zero in the symmetric case. In this case equality holds when $x = (z,-s)$ and $y_0 = (z,s)$ for some $z \in A$ and $s \in [-a,a]$. Thus the second derivatives of Z which vanish are those in the directions $\dot{x} = \dot{y} = (e,0)$ or $\dot{x} = (0,-1)$, $\dot{y} = (0,1)$. We choose an orthonormal basis where $e_n = \frac{y_0 - x_0}{|y_0 - x_0|}$, and compute the corresponding second derivatives:

$$\left.\frac{d^2}{ds^2} Z(x_0 + s e_i, y_0 + s e_i)\right|_{s=0} = u_{ii}(y_0,t_0) - u_{ii}(x_0,t_0) \leq 0;$$

and

$$\left.\frac{d^2}{ds^2} Z(x_0 + s e_n, y_0 - s e_n)\right|_{s=0} = u_{nn}(y_0,t_0) - u_{nn}(x_0,t_0) - 2\varphi'' \leq 0.$$

Adding these gives $\Delta u(y_0,t_0) - \Delta u(x_0,t_0) - 2\varphi'' \leq 0$, which combines with the time inequality to give $\varphi_t \leq \varphi''$, as required. □

The result we just proved has a very nice consequence:

Theorem (The Payne-Weinberger inequality)

The first nontrivial eigenvalue λ_1 of the Neumann Laplacian on a convex domain Ω in $\mathbb{R}^n$ satisfies $\lambda_1 \geq \frac{\pi^2}{D^2}$. Equality holds in the limit of examples of the form $\Omega = A \times I$ where the diameter of A approaches zero.

Proof. Let $u_1(x)$ be the corresponding eigenfunction. Then $u(x,t) = \mathrm{e}^{-\lambda_1 t} u_1(x)$ is a solution of the Neumann heat equation on Ω. Let $\omega : [0, D/2] \times [0, \infty)$ be the corresponding modulus of continuity. Then $\omega(s, 0) \leq C \sin\left(\frac{\pi s}{D}\right)$ for some large C, since u_1 is smooth. The previous result implies $\omega(s,t) \leq C\mathrm{e}^{-\frac{\pi^2}{D^2}t} \sin\left(\frac{\pi s}{D}\right)$ for all $s \in [0, D/2]$ and $t \geq 0$. In particular this implies $\operatorname{osc}(u_1)\mathrm{e}^{-\lambda_1 t} = \operatorname{osc}(u(.,t)) \leq C\mathrm{e}^{-\frac{\pi^2}{D^2}t}$. Taking $t \to \infty$ implies $\lambda_1 \geq \frac{\pi^2}{D^2}$. □

The heat equation on a Riemannian manifold

We can carry out the same argument for the heat equation $\frac{\partial u}{\partial t} = \Delta u$ on a compact Riemannian manifold, by taking the following definition of the modulus of continuity:

$$\omega(s,t) := \sup\left\{\frac{u(y,t)-u(x,t)}{2} : \frac{d(x,y)}{2} = s\right\}$$

where $d(x,y)$ is the Riemannian distance from x to y defined by minimising lengths of curves joining x and y. The result in this case is the following:

Proposition

Suppose (M,g) is a compact Riemannian manifold with diameter $D = \sup\{d(x,y) : x,y \in M\}$ and Ricci curvature bounded below: $Rc \geq (n-1)Kg$. If u satisfies the heat equation on M, then ω satisfies

$$\omega_t(s,t) \leq \omega''(s,t) + (n-1)T_K(s)\omega'(s,t)$$

in the viscosity sense. Equality holds in the case that M is a warped product $M = \Sigma^{n-1} \times [-D/2, D/2]$, with $g = ds^2 + C_K(s)\bar{g}$ for some metric $\bar{g}$ on Σ, and u is a function of s and t only.

The proof uses inequalities on the Hessian of the distance function on $M \times M$, given by the Ricci curvature lower bound (similar to the Laplacian comparison theorem), and is otherwise carried out by making choices of second variation inequalities for which equality holds in the model case.

An immediate consequence is the following sharp eigenvalue lower bound, which was previously known using gradient estimates on eigenfunctions through work of Li and Yau, Zhong and Yang, Kröger, and Bakry and Qian, and using probabilistic arguments by Mu-Fa Chen and Fengyu Wang:

Proposition

The first nontrivial eigenvalue λ_1 of the (Neumann) Laplacian on a compact manifold M^n with diameter bounded by D, Ricci curvature bounded below by $(n-1)Kg$, and boundary convex (if non-empty) satisfies

$$\lambda_1 \geq \bar{\lambda}_1(n,K,D)$$

where $\bar{\lambda}_1(n,K,D)$ is the first eigenvalue of the one-dimensional heat equation

$$f_t = f'' + (n-1)T_K f'$$

with Neumann boundary conditions on $[-D/2, D/2]$. Equality holds in the limit of warped product examples $\Sigma \times I$ as the diameter of Σ approaches zero.

Particular cases: $\bar{\lambda}_1(n,0,D) = \frac{\pi^2}{D^2}$ (Li-Yau, Zhong-Yang); $\bar{\lambda}(n,1,\pi) = n$ (Lichnerowicz).

Now I want to return to the Euclidean setting to discuss the corresponding problem for the Dirichlet Laplacian, which is the 'fundamental gap conjecture' of van den Berg, Yau and Ashbaugh-Benguria:

Theorem (A.-Clutterbuck)

Let Ω be a convex domain on diameter D in $\mathbb{R}^n$, and V a convex function on Ω. Let λ_0 and λ_1 be the first two eigenvalues of the corresponding Schrödinger operator with Dirichlet boundary condition:

$$\begin{cases} \Delta\varphi_i + V\varphi_i + \lambda_i\varphi_i = 0, & \text{on } \Omega; \\ \varphi_i = 0 & \text{on } \partial\Omega, \end{cases}$$

for $i = 0, 1$, with $\varphi_0 > 0$ on Ω. Then

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}.$$

Equality holds in the limit of domains of the form $A \times I$ as the diameter of A approaches zero, with zero potential.

Ratios of positive solutions to the Dirichlet heat equation

The first part of the proof is similar to what I showed you for the proof of the Payne-Weinberger inequality, using the following nice observation: Suppose that u_0 and u_1 are two solutions of the corresponding heat equation:

$$\begin{cases} \frac{\partial u_i}{\partial t} = \Delta u_i + V u_i & \text{on } \Omega \times [0, \infty); \\ u_i = 0 & \text{on } \partial\Omega \times [0, \infty), \end{cases}$$

for $i = 0, 1$, and assume that $u_0 > 0$ on $\Omega \times [0, \infty)$. Then the ratio $v = \frac{u_1}{u_0}$ satisfies the following Neumann heat equation with drift:

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v + 2D\log u_0 \cdot Dv & \text{on } \Omega \times [0, \infty) \\ D_\nu v = 0 & \text{on } \partial\Omega \times [0, \infty). \end{cases}$$

The idea is now to control the modulus of continuity of v. When we do this, we need to show that

$$Z(x, y, t) = v(y, t) - v(x, t) - 2\varphi\left(\frac{|y - x|}{2}, t\right)$$

stays non-positive, provided φ is a solution of some suitable equation. The only difference from before is in the time derivative, where we get an extra term:

$$\frac{\partial Z}{\partial t} = \Delta u(y, t) - \Delta u(x, t) + 2\left(D\log u_0 \cdot Dv\big|_{(y_0, t_0)} - D\log u_0 \cdot Dv\big|_{(x_0, t_0)}\right).$$

$$\frac{\partial Z}{\partial t} = \Delta u(y_0,t_0) - \Delta u(x_0,t_0) + 2\left(D\log u_0 \cdot Dv\big|_{(y_0,t_0)} - D\log u_0 \cdot Dv\big|_{(x_0,t_0)}\right).$$

The last term simplifies a little by using the first derivative conditions:

$$DZ\Big|_{(x_0,y_0,t_0)}(e_1,e_2) = Dv\big|_{(y_0,t_0)}(e_2) - Dv\big|_{(x_0,t_0)}(e_1) - \varphi' \frac{y_0-x_0}{|y_0-x_0|}\cdot(e_2-e_1),$$

from which it follows that $Dv|_{(x_0,t_0)} = Dv|_{(y_0,t_0)} = \varphi' \frac{y_0-x_0}{|y_0-x_0|}$. This gives

$$\frac{\partial Z}{\partial t} = \Delta u(y,t) - \Delta u(x,t) + 2\varphi'\left(D\log u_0\big|_{(y_0,t_0)} - D\log u_0\big|_{(x_0,t_0)}\right)\cdot\frac{y_0-x_0}{|y_0-x_0|}.$$

So the extra piece of information we need is control on this extra term. It was proved by Brascamp and Lieb that the first eigenfunction φ_0 has concave logarithm, and this implies the extra term is non-positive if we make the choice $u_0 = \varphi_0 e^{-\lambda_0 t}$, so we can discard it and deduce as before that ω is a subsolution of the 1D heat equation. By taking $u_i = \varphi_i e^{-\lambda_i t}$ for $i = 0,1$, this in turn implies the estimate $\lambda_1 - \lambda_0 \geq \frac{\pi^2}{D^2}$ (proved previously by Yu and Zhong using gradient estimates, building on the initial groundbreaking work of Singer-Wong-Yau-Yau).

To get a sharp estimate on the gap, we need a sharp estimate on the log-concavity of the first eigenfunction.

Here is the sharp log-concavity estimate that we prove:

Proposition

Let Ω be a convex domain of diameter D, and φ_0 the first Dirichlet eigenfunction:

$$\begin{cases} \Delta\varphi_0 + V\varphi_0 + \lambda_0\varphi_0 = 0 & \text{on } \Omega; \\ \varphi_0 = 0 & \text{on } \partial\Omega; \\ \varphi_0 > 0 & \text{on } \Omega, \end{cases}$$

with V convex. Then

$$(D\log\varphi_0(y,t) - D\log\varphi_0(x,t)) \cdot \frac{y-x}{|y-x|} \leq -\frac{2\pi}{D}\tan\left(\frac{\pi|y-x|}{2D}\right)$$

for all $x \neq y$ in Ω.

Note that equality holds when Ω is an interval $[-D/2, D/2]$, $V = 0$, and $x + y = 0$.
Proof: Two-point maximum principle applied to the difference in the claimed inequality, using second derivative terms chosen to give equality in the model case $A \times [-D/2, D/2]$. □

Given the estimate of the last proposition, we deduce that (assuming $u_0 = \varphi_0 e^{-\lambda_0 t}$) the modulus of continuity satisfies

$$\omega_t \le \omega'' - 2\frac{\pi}{D}\tan\left(\frac{\pi s}{D}\right)\omega'$$

in the viscosity sense, and this has the explicit solution $\sin(\frac{\pi s}{D})e^{-\frac{3\pi^2}{D^2}t}$ coming from the $1D$ case. This gives the sharp estimate $\lambda_1 - \lambda_0 \ge \frac{3\pi^2}{D^2}$.